Dynamics of nonlinear resonances in Hamiltonian systems

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Abstract – It is well known that the dynamics of a Hamiltonian system depends crucially on whether or not it possesses nonlinear resonances. In the generic case, the set of nonlinear resonances consists of independent clusters of resonantly interacting modes, described by a few low-dimensional dynamical systems. We formulate and prove a new theorem on integrability which allows us to show that most frequently met clusters are described by integrable dynamical systems. We argue that construction of clusters can be used as the base for the Clipping method, substantially more effective for these systems than the Galerkin method. The results can be used directly for systems with cubic Hamiltonian.

Introduction. – The very special role of resonant solutions of nonlinear ordinary differential equations (ODEs) has been first investigated by Poincaré at the end of the 19th century. Poincaré proved that if a nonlinear ODE has no resonance solutions, then it can be linearized by an invertible change of variables (for details see [1] and references therein). This simplifies both analytical and numerical investigations of the original nonlinear equation, allows for the introduction of corresponding normal forms of ODEs, etc. In the middle of the 20th century, Poincaré's approach has been generalized to the case of nonlinear partial differential equations (PDEs) yielding what is nowadays known as KAM theory (*e.g.* [2]). This theory allows us to transform a nonlinear dispersive PDE into a Hamiltonian equation of motion in Fourier space [3],

$$i\,\dot{a}_{\mathbf{k}} = \partial\mathcal{H}/\partial a_{\mathbf{k}}^*, \qquad (1)$$

where $a_{\mathbf{k}}$ is the amplitude of the Fourier mode corresponding to the wave vector $\mathbf{k}$ and the Hamiltonian $\mathcal{H}$ is represented as an expansion in powers $\mathcal{H}_j$ which are proportional to the product of j amplitudes $a_{\mathbf{k}}$. In this letter we are going to consider expansions of Hamiltonians up to third order in wave amplitude, *i.e.* a cubic Hamiltonian of the form

$$\mathcal{H}_3 = \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}} V_{23}^1 a_1^* a_2 a_3 \delta_{23}^1 + \text{c.c.},$$

where for brevity we introduced the notation $a_j \equiv a_{\mathbf{k}_j}$ and $\delta_{23}^1 \equiv \delta(\mathbf{k_1} - \mathbf{k_2} - \mathbf{k_3})$ is the Kronecker symbol. If $\mathcal{H}_3 \neq 0$, three-wave process is dominant and the main contribution to the nonlinear evolution comes from the waves satisfying the following resonance conditions:

$$\begin{cases} \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3) = \Omega, \\ \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = 0, \end{cases} \qquad (2)$$

where $\omega(\mathbf{k})$ is a dispersion relation for the linear wave frequency and $\Omega \geqslant 0$ is called resonance width.

If $\Omega > 0$, we denote $a_{\mathbf{k}}$ as $A_{\mathbf{k}}$ and the equation of motion (1) turns into

$$i\dot{A}_{\mathbf{k}} = \omega_{\mathbf{k}} A_{\mathbf{k}} + \sum_{\mathbf{k}_1,\mathbf{k}_2} V_{12}^{\mathbf{k}} A_1 A_2 \delta_{12}^{\mathbf{k}} + 2 V_{\mathbf{k}2}^{1\,*} A_1 A_2^* \delta_{\mathbf{k}2}^1. \quad (3)$$

If $\Omega = 0$, we denote $a_{\mathbf{k}}$ as $B_{\mathbf{k}}$ and the equation of motion (1) turns into

$$\begin{aligned} i\dot{B}_{\mathbf{k}} = \sum_{\mathbf{k}_1,\mathbf{k}_2} \big(& V_{12}^{\mathbf{k}} B_1 B_2 \delta_{12}^{\mathbf{k}} \delta(\omega_{\mathbf{k}} - \omega_1 - \omega_2) \\ & + 2 V_{\mathbf{k}2}^{1\,*} B_1 B_2^* \delta_{\mathbf{k}2}^1 \delta(\omega_1 - \omega_{\mathbf{k}} - \omega_2) \big). \end{aligned} \qquad (4)$$

The co-existence of these two substantially different types of wave interactions, described by eqs. (3) and (4), has been observed in numerical simulations [4] and proven analytically in the frame of the kinematic two-layer model of laminated turbulence [5]. Dynamics of the layer (3) is

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described by wave kinetic equations and is well studied [3]. Dynamics of the layer (4) is practically not studied though a lot of preliminary results are already known. Namely, the layer (4) is described by a few independent wave clusters formed by the waves which are in exact nonlinear resonance [6]. The solutions of (2), corresponding to $\Omega = 0$, can be computed by the specially developed q-class method [7] and the general form of dynamical systems describing resonant clusters can also be found algorithmically [8], as well as coefficients of dynamical systems [9]. Moreover, as it was demonstrated in [10] (numerically) and in [11] (analytically), these clusters "survive" for small enough but non-zero Ω, which corresponds to the accuracy of numerical simulations or laboratory experiments. The main goal of this letter is to study the dynamics of the most frequently met resonant clusters.

Clusters. – In this letter we present some analytical and numerical results for the three most commonly met dynamical systems corresponding to non-isomorphic clusters of nonlinear resonances —*a triad*, *a kite*, and *a butterfly* consisting of 3, 4 and 5 complex variables correspondingly.

The dynamical system for *a triad* has the standard Manley-Rowe form

$$\dot{B}_1 = ZB_2^*B_3, \quad \dot{B}_2 = ZB_1^*B_3, \quad \dot{B}_3 = -ZB_1B_2, \quad (5)$$

where Z is a constant called interaction coefficient.

A kite consists of two triads a and b, with wave amplitudes B_{ja}, B_{jb}, $j = 1, 2, 3$, connected *via* two common modes. Analogously to [12], one can point out 4 types of kites according to the properties of connecting modes. For our considerations, this is not important: the general method to study integrability of kites will be the same. For the concreteness of presentation, in this letter *a kite* with $B_{1a} = B_{1b}(\equiv B_1)$ and $B_{2a} = B_{2b}(\equiv B_2)$ has been chosen:

$$\begin{cases} \dot{B}_1 = B_2^*(Z_a B_{3a} + Z_b B_{3b}), \\ \dot{B}_2 = B_1^*(Z_a B_{3a} + Z_b B_{3b}), \\ \dot{B}_{3a} = -Z_a B_1 B_2, \quad \dot{B}_{3b} = -Z_b B_1 B_2. \end{cases} \quad (6)$$

A butterfly consists of two triads a and b, with wave amplitudes B_{ja}, B_{jb}, $j = 1, 2, 3$, connected *via* one common mode. As was shown in [12], there exist 3 different types of butterflies, according to the choice of the connecting mode. Let us take, for instance, $B_{1a} = B_{1b}(\equiv B_1)$. The corresponding dynamical system is then as follows:

$$\begin{cases} \dot{B}_1 = Z_a B_{2a}^* B_{3a} + Z_b B_{2b}^* B_{3b}, \\ \dot{B}_{2a} = Z_a B_1^* B_{3a}, \quad \dot{B}_{2b} = Z_b B_1^* B_{3b}, \\ \dot{B}_{3a} = -Z_a B_1 B_{2a}, \quad \dot{B}_{3b} = -Z_b B_1 B_{2b}. \end{cases} \quad (7)$$

Integrability of resonance clusters. – From here on, general notations and terminology will follow Olver's book [13]. We use hereafter Einstein convention on repeated indices and $f_{,i} \equiv \partial f / \partial x^i$. Consider a general

n-dimensional system of autonomous evolution equations of the form

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}(t) = \Delta^i(x^j(t)), \qquad i = 1, \ldots, n. \quad (8)$$

Any scalar function $f(x^i, t)$ that satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\left(f(x^i(t), t)\right) = \frac{\partial}{\partial t}f + \Delta^i f_{,i} = 0$ is called *a conservation law* in [13]. It is easy to see that this definition gives us two types of conservation laws. The first type is the standard notion used in classical physics: the conservation law is of the form $f(x^i)$, *i.e.* it does not depend explicitly on time. The second type looks more like a mathematical trick: it is of the form $f(x^i, t)$, where the time dependence is explicit. In this letter we will be interested in both types of conservation law because they are both physically important. To keep in mind the difference between these two types of conservation laws, we will call the first type just a conservation law (CL), and we will call the second type a *dynamical invariant*.

We say that syst. (8) is *integrable* if there are n functionally independent dynamical invariants. Obviously, if syst. (8) possesses $(n-1)$ functionally independent CLs, then it is constrained to move along a 1-dimensional manifold, and the way it moves is dictated by 1 dynamical invariant. This dynamical invariant can be obtained from the knowledge of the $(n-1)$ CLs and the explicit form of syst. (8), *i.e.* syst. (8) is integrable then. It follows from the theorem below that in many cases the knowledge of only $(n-2)$ CLs is enough for the integrability of syst. (8).

Theorem on $(n-2)$-integrability. Let us assume that syst. (8) *possesses a standard Liouville volume density*

$$\rho(x^i) : (\rho\Delta^i)_{,i} = 0,$$

and $(n-2)$ functionally independent CLs, $H^1, \ldots, H^{n-2}$. Then a new CL in quadratures can be constructed, which is functionally independent of the original ones, and therefore the system is integrable.

The (lengthy) complete proof follows from the existence of a Poisson bracket for the original syst. (8) under the assumptions of the theorem and is an extension of the general approach used in [14] for three-dimensional first-order autonomous equations. The proof is constructive and allows us to find the explicit form of a new CL for dynamical systems of the form (5), (6), (7), etc.

For simplicity of presentation, the proof is given here for the case $n = 2$. The dynamical system is just 2-dimensional and we will write it as a vector $(\Delta^1, \Delta^2)^T$. The theorem requires the existence of a standard Liouville volume density $\rho(x^1, x^2)$ satisfying

$$(\rho\Delta^1)_{,1} + (\rho\Delta^2)_{,2} = 0, \quad (9)$$

and does not require the knowledge of conservation laws. The theorem allows us to find by quadratures a conservation law $H(x^1, x^2)$ for the system. The procedure goes along the lines of constructing a Hamiltonian

structure: one has to solve for the Hamiltonian $H(x^1, x^2)$ the following equation:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{,1} \\ H_{,2} \end{pmatrix} = \begin{pmatrix} \rho \Delta^1 \\ \rho \Delta^2 \end{pmatrix}. \qquad (10)$$

This equation is easy to solve for the gradient of H and the solution is: $H_{,1} = -\rho \Delta^2$, $H_{,2} = \rho \Delta^1$. Because the solution is obtained for the gradient of H, there is a compatibility condition: $H_{,1,2} = H_{,2,1}$. This condition is equivalent to eq. (9) and is by assumption fulfilled. H can be obtained by quadratures. Now it is easy to see that H is a CL: $\Delta^i H_{,i} = 0$. This finishes the proof for the case $n = 2$. The proof for $n > 2$ is along the same lines but requires the introduction of differential forms of order higher than two and is omitted here.

The statement of the above theorem is implicitly of a local character, in the sense that the obtained CLs are defined in a neighborhood of a point in phase space about which the system is considered. Considerations of global character are related to the concept of superintegrability [15] and are outside the scope of this paper.

In the examples below, we always need to eliminate so-called slave phases, which corresponds to the well-known order reduction in Hamiltonian systems [16]. The number n used below corresponds to the effective number of degrees of freedom after this reduction has been performed.

Integrability of a triad, dynamical system (5), is a well-known fact (e.g. [17]) and its two conservation laws are

$$I_{23} = |B_2|^2 + |B_3|^2, \ I_{13} = |B_1|^2 + |B_3|^2.$$

System (5) has been used for a preliminary check of our method; in this case $n = 4$. The method can thus be applied and we obtain the following CL:

$$I_T = \mathrm{Im}(B_1 B_2 B_3^*),$$

together with the time-dependent dynamical invariant of the form

$$S_0 = Z\,t - \frac{F\left(\arcsin\left(\left(\frac{R_3 - v}{R_3 - R_2}\right)^{1/2}\right), \left(\frac{R_3 - R_2}{R_3 - R_1}\right)^{1/2}\right)}{2^{1/2}(R_3 - R_1)^{1/2}(I_{13}^2 - I_{13}I_{23} + I_{23}^2)^{1/4}}.$$

Here F is the elliptic integral of the first kind, $R_1 < R_2 < R_3$ are the three real roots of the polynomial

$$x^3 + x^2 = 2/27 - (27 I_T^2 -$$

$$(I_{13} + I_{23})(I_{13} - 2I_{23})(I_{23} - 2I_{13}))/27(I_{13}^2 - I_{13}I_{23} + I_{23}^2)^{3/2}$$

and

$$v = |B_1|^2 - (2I_{13} - I_{23} + (I_{13}^2 - I_{13}I_{23} + I_{23}^2)^{1/2})/3$$

is always within the interval $[R_2, R_3]$ which contains the zero. Notice that the period of the motion (i.e. the time

it takes $v(t)$ to go from R_2 to R_3 and then back to R_2) could be easily computed from the above equation.

A kite, dynamical system (6), is also an integrable system. Indeed, after reduction of slave variables the system corresponds to $n = 6$ and has 5 CLs (2 linear, 2 quadratic, 1 cubic):

$$\begin{cases} L_R = \mathrm{Re}(Z_b B_{3a} - Z_a B_{3b}), \ L_I = \mathrm{Im}(Z_b B_{3a} - Z_a B_{3b}), \\ I_{1ab} = |B_1|^2 + |B_{3a}|^2 + |B_{3b}|^2, \\ I_{2ab} = |B_2|^2 + |B_{3a}|^2 + |B_{3b}|^2, \\ I_K = \mathrm{Im}(B_1 B_2 (Z_a B_{3a}^* + Z_b B_{3b}^*)), \end{cases}$$

with a dynamical invariant that is essentially the same as for a triad, S_0, after replacing $Z = Z_a + Z_b$, $I_T = I_K(Z_a^2 + Z_b^2)/Z^3$, $I_{13} = I_{1ab}(Z_a^2 + Z_b^2)/Z^2 - (L_R^2 + L_I^2)/Z^2$, $I_{23} = I_{2ab}(Z_a^2 + Z_b^2)/Z^2 - (L_R^2 + L_I^2)/Z^2$.

The dynamics of a butterfly is governed by eqs. (7) and its 4 CLs (3 quadratic and 1 cubic) can easily be obtained:

$$\begin{cases} I_{23a} = |B_{2a}|^2 + |B_{3a}|^2, \quad I_{23b} = |B_{2b}|^2 + |B_{3b}|^2, \\ I_{ab} = |B_1|^2 + |B_{3a}|^2 + |B_{3b}|^2, \\ I_0 = \mathrm{Im}(Z_a B_1 B_{2a} B_{3a}^* + Z_b B_1 B_{2b} B_{3b}^*), \end{cases} \qquad (11)$$

while a Liouville volume density is $\rho = 1$. Notice that all cubic CLs are canonical Hamiltonians for the respective triad, kite and butterfly systems. From now on we consider the butterfly case when no amplitude is identically zero; otherwise the system would become integrable.

The use of standard amplitude-phase representation $B_j = C_j \exp(i\theta_j)$ of the complex amplitudes B_j in terms of real amplitudes C_j and phases θ_j shows immediately that only two phase combinations are important:

$$\varphi_a = \theta_{1a} + \theta_{2a} - \theta_{3a}, \quad \varphi_b = \theta_{1b} + \theta_{2b} - \theta_{3b},$$

called a- and b-triad phases (with the requirement $\theta_{1a} = \theta_{1b}$ which corresponds to the choice of the connecting mode $B_{1a} = B_{1b}$.) This reduces five complex equations (7) to only four real ones:

$$\frac{\mathrm{d}C_{3a}}{\mathrm{d}t} = -Z_a C_1 C_{2a} \cos \varphi_a, \qquad (12)$$

$$\frac{\mathrm{d}C_{3b}}{\mathrm{d}t} = -Z_b C_1 C_{2b} \cos \varphi_b, \qquad (13)$$

$$\frac{\mathrm{d}\varphi_a}{\mathrm{d}t} = Z_a C_1 \left(\frac{C_{2a}}{C_{3a}} - \frac{C_{3a}}{C_{2a}}\right) \sin \varphi_a - \frac{I_0}{(C_1)^2}, \qquad (14)$$

$$\frac{\mathrm{d}\varphi_b}{\mathrm{d}t} = Z_b C_1 \left(\frac{C_{2b}}{C_{3b}} - \frac{C_{3b}}{C_{2b}}\right) \sin \varphi_b - \frac{I_0}{(C_1)^2}. \qquad (15)$$

The cubic CL reads

$$I_0 = C_1 \left(Z_a C_{2a} C_{3a} \sin \varphi_a + Z_b C_{2b} C_{3b} \sin \varphi_b\right) \qquad (16)$$

in terms of the amplitudes and phases. This means that the dynamics of a butterfly cluster is, in the generic case, confined to a 3-dimensional manifold. Below we regard a few particular cases in which syst. (7) is integrable.
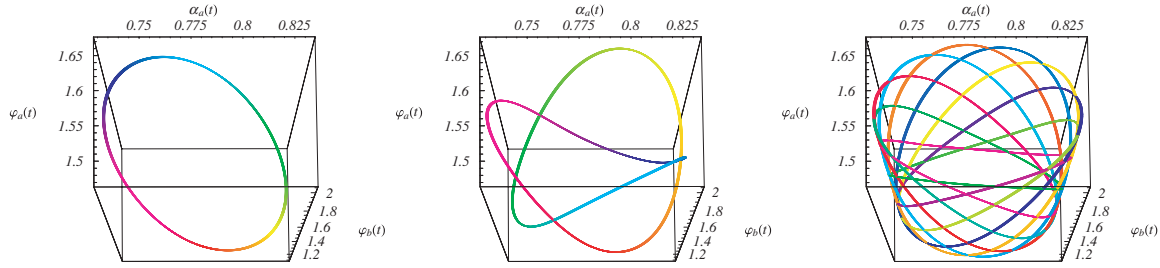
Fig. 1: (Color online) To facilitate view, color hue of the plot is a linear function of time t, varying from 0 to 1 as t runs through one quasi-period. Left panel: $Z_a = Z_b = 10/100$ (integrable case), $I_{ab} \approx 2.1608$. Middle panel: $Z_a = 8/100$, $Z_b = 12/100$, $I_{ab} \approx 2.2088$. Right panel: $Z_a = 9/100$, $Z_b = 11/100$, $I_{ab} \approx 2.1846$.

Example 1: real amplitudes, $\varphi_a = \varphi_b = 0$. In this case the Hamiltonian I_0 becomes identically zero, while Liouville density in coordinates C_{3a}, C_{3b} is $\rho(C_{3a}, C_{3b}) = 1/C_1 C_{2a} C_{2b}$. So in this case the equations for the unknown CL $H(C_{3a}, C_{3b})$ are

$$\rho \Delta^{C_{3a}} = -\frac{Z_a}{C_{2b}} = \frac{\partial}{\partial C_{3b}} H, \quad \rho \Delta^{C_{3b}} = -\frac{Z_b}{C_{2a}} = -\frac{\partial}{\partial C_{3a}} H,$$

and from eqs. (11) we readily obtain

$$H(C_{3a}, C_{3b}) = Z_b \arctan\left(C_{3a}/C_{2a}\right) - Z_a \arctan\left(C_{3b}/C_{2b}\right),$$

i.e. syst. (7) is integrable in this case. Of course, this case is degenerate for $I_0 \equiv 0$ yields no constraint on the remaining independent variables C_{3a}, C_{3b} satisfying eqs. (12), (13).

General change of coordinates. Going back to eqs. (12)–(16), we can jump from this degenerate case to a more generic case by defining new coordinates which are suggested by $H(C_{3a}, C_{3b})$. The new coordinates are to replace the amplitudes C_{3a}, C_{3b}:

$$\alpha_a = \arctan\left(C_{3a}/C_{2a}\right), \quad \alpha_b = \arctan\left(C_{3b}/C_{2b}\right).$$

We choose the inverse transformation to be

$$\begin{cases} C_{2a} = \sqrt{I_{23a}}\cos(\alpha_a), & C_{3a} = \sqrt{I_{23a}}\sin(\alpha_a), \\ C_{2b} = \sqrt{I_{23b}}\cos(\alpha_b), & C_{3b} = \sqrt{I_{23b}}\sin(\alpha_b), \end{cases} \quad (17)$$

so that the domain for the new variables is $0 < \alpha_a < \pi/2, 0 < \alpha_b < \pi/2$. for these new coordinates, the evolution equations simplify enormously:

$$\begin{cases} \dfrac{\mathrm{d}\alpha_a}{\mathrm{d}t} = -Z_a C_1 \cos\varphi_a, \quad \dfrac{\mathrm{d}\alpha_b}{\mathrm{d}t} = -Z_b C_1 \cos\varphi_b, \\[2mm] \dfrac{\mathrm{d}\varphi_a}{\mathrm{d}t} = Z_a C_1 \left(\cot\alpha_a - \tan\alpha_a\right)\sin\varphi_a - \dfrac{I_0}{(C_1)^2}, \\[2mm] \dfrac{\mathrm{d}\varphi_b}{\mathrm{d}t} = Z_b C_1 \left(\cot\alpha_b - \tan\alpha_b\right)\sin\varphi_b - \dfrac{I_0}{(C_1)^2}, \end{cases} \quad (18)$$

where the amplitude $C_1 > 0$ is obtained using eqs. (11):

$$C_1 = \sqrt{I_{ab} - I_{23a}\sin^2\alpha_a - I_{23b}\sin^2\alpha_b} \quad (19)$$

and the cubic CL is now

$$I_0 = \frac{C_1}{2}\left(Z_a I_{23a}\sin(2\alpha_a)\sin(\varphi_a) + Z_b I_{23b}\sin(2\alpha_b)\sin(\varphi_b)\right). \quad (20)$$

Equations (18)–(20) represent the final form of our 3-dimensional general system.

Example 2: complex amplitudes, $I_0 = 0$. Here, we just impose the condition $I_0 = 0$ but the phases are otherwise arbitrary: this case is therefore not degenerate anymore and we have a 3-dimensional system which requires the existence of only 1 CL in order to be integrable: a CL is $A_a = \sin(2\alpha_a)\sin(\varphi_a)$, which can be deduced from eqs. (18). Making use of the theorem, one can find another CL for this case: $H_{new}(C_{3a}, C_{3b}) = (1 + Z_b/Z_a)\arccos\left(\frac{\cos 2\alpha_a}{\sqrt{1 - A_a^2}}\right) - (1 + Z_a/Z_b)\arccos\left(\frac{\cos 2\alpha_b}{\sqrt{1 - A_b^2}}\right)$. Obviously A_a and H_{new} are functionally independent, *i.e.* the case $I_0 = 0$ is integrable.

Example 3: complex amplitudes, $Z_a = Z_b$. In this case a new CL has the form

$$\frac{I_0^2}{Z_a}E = C_{2a}^2 C_{3a}^2 + C_{2b}^2 C_{3b}^2 + 2C_{2a}C_{3a}C_{2b}C_{3b}\cos(\varphi_a - \varphi_b) - C_1^2(C_1^2 - C_{2a}^2 + C_{3a}^2 - C_{2b}^2 + C_{3b}^2), \quad (21)$$

which is functionally independent of the other known constants of motion. Therefore, according to the theorem the case $Z_b = Z_a$ is integrable.

The numerical scheme is programmed in *Mathematica* with *stiffness-switching* method in single precision. For arbitrary Z_a, Z_b the scheme has been checked by computing I_0 from eq. (20) at all consequent time steps; there is no noticeable change of I_0 up to machine precision.

Numerical simulations. – To investigate the general behavior of a butterfly cluster with $Z_a \neq Z_b$, we integrated directly eqs. (18), (19) with I_0 computed from eq. (20) evaluated at $t = 0$ and used to check numerical scheme afterwards. Some results of the simulations with syst. (18) are presented in fig. 1. Initial conditions $\alpha_a(0) = 78/100$, $\alpha_b(0) = 60/100$, $\varphi_a(0) = 147/100$, $\varphi_b(0) = 127/100$ and values of the constants of motion $I_0 = 11/2000$,
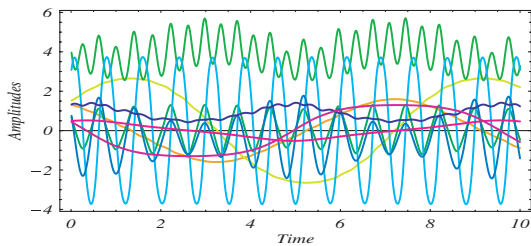
Fig. 2: (Color online) Chain of 4 triads, real amplitudes for all 9 modes. Time and amplitudes in non-dimensioned units.

$I_{23a} = 4/100$, $I_{23b} = 4/100$ are the same for all three parts of fig. 1. Three-dimensional parametric plots are shown in space $(\alpha_a, \varphi_a, \varphi_b)$ with color hue depending on time, so the plots are effectively 4D. The main goal of this series of numerical simulations was to study changes in the dynamics of a butterfly cluster according to the magnitude of the ratio $\zeta = Z_a/Z_b$. In fig. 1, left panel $\zeta_l = 1$, middle panel $\zeta_m = 2/3$ and right panel $\zeta_r = 9/11$. As was shown above, the case $\zeta_l = 1$ is integrable, and one can see in the left panel a seemingly closed trajectory with quasi-period $T_l \approx 21.7$. A closer look shows that this trajectory is not closed, rather the orbit precesses, which is a generic feature of integrable systems that are not superintegrable.

Rational ζ_m, ζ_r produce again what appear to be periodic motions and seemingly closed trajectories with quasi-periods $T_m \approx 53$ and $T_r \approx 215$ correspondingly. Again a closer look shows that the orbits precess but we do not know in this case if the precession is associated to integrability or not.

A few dozen of simulations made with different rational ratios $\zeta = Z_a/Z_b$ show that quasi-periodicity depends on —or is even defined by— the commensurability of the coefficients Z_a and Z_b. Figure 1 shows that $\zeta_m = 2/3$ gives 2 spikes in one direction and 3 spikes in the perpendicular direction, while $\zeta_r = 9/11$ gives 9 and 11 spikes correspondingly; and so on.

Some preliminary series of simulations have been performed in order to study chains of triads with connection types as in (7). The maximum number of triads in a chain was 8, which corresponds to 17 modes. For our numerical simulations, resonant clusters of spherical Rossby waves were taken, with initial (non-dimensioned) energies of the order of measured atmospheric data, as in [18]. The dynamical system for computations was taken in the original variables B_j. The case of real amplitudes is shown in fig. 2: all resonant modes behave quasi-periodically. Non-dimensioned units for time and amplitudes were chosen to illustrate clearly the characteristic behavior of the amplitudes.

Discussion. – Our analysis and general mathematical results [6] on resonant clusters are valid for arbitrary Hamiltonian $\mathcal{H}_j, j \geqslant 3$, though computation of clusters in the case $j > 3$ is more involved. To keep track of the evolution of triads and kites (which are integrable clusters)

as well as the evolution of bigger clusters (butterflies and so on), can be regarded as an alternative method to integrate numerically a given nonlinear PDE. This method is known as the clipping method (CM), introduced in [19] in order to deal with evolutionary dispersive nonlinear PDEs. Using computer methods [7], one can construct all resonant clusters in the chosen spectral domain and perform numerical simulations of only the dynamical systems corresponding to resonant clusters. As for each of the non-resonant modes, their energies are approximately constant during many periods of energy exchange of the resonant modes, so these non-resonant modes can be discarded.

Example for atmospheric planetary waves is given in [10]: wave vectors **k**, as well as frequencies ω are functions of two integer parameters m and n which can be regarded as labels of corresponding Fourier harmonics. If one takes $m \leqslant n \leqslant 21$, the overall number of Fourier harmonics is 221, among them 50 are resonant and divided into 8 clusters: 4 triads, 3 butterflies and one cluster consisting of 13 resonant modes. Another example is given in [8] for ocean planetary motions and $m, n \leqslant 50$. The overall number of Fourier harmonics is 2500, among them 128 resonant modes divided into 28 clusters: 18 triads, 2 butterflies and 8 clusters of a more complicated structure; maximal cluster consists of 13 modes.

The clipping method has at least three advantages compared to Galerkin truncation: i) Numerical schemes in CM can be truncated at a substantially higher number of Fourier harmonics than in GM, depending not on the computer facilities but on some physically relevant parameters (say, dissipation range of wave vectors **k**). ii) Most of the resulting dynamical systems are triads which are integrable. iii) The solutions obtained from the integrable cases could be used to parameterize the numerical solutions of non-integrable systems found for bigger clusters. This work is in progress.

Last but not least. Even for one specific PDE, it is a highly non-trivial task to prove that Galerkin truncation is a Hamiltonian system and to construct an additional conserved quantity [20]. The clipping method combined with the constructive procedure based on the theorem on $(n - 2)$-integrability, allows us to produce physically relevant dynamical systems and to find additional conservation laws systematically, for a wide class of evolutionary dispersive nonlinear PDEs.

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