Determination of the complete set of statically balanced planar four-bar mechanisms*

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Abstract

In this paper, we present a new method to determine the complete set of statically balanced planar four-bar mechanisms. We formulate the kinematic constraints and the static balancing constraints as algebraic equations over real and complex variables. This leads to the problem of factorization of Laurent polynomials which can be solved using Newton polytopes and Minkowski sums. The result of this process is a set of necessary and sufficient conditions for statically balanced four-bar mechanisms.

1 Introduction

A *statically balanced* mechanism is defined as a mechanism in which the weight of the links does not produce any torque (or force) at the actuators under static conditions, for any configuration of the mechanism. This condition is also referred to as *gravity compensation*. Many statically balanced mechanisms based on the use of counterweights, springs and sometimes cams and/or pulleys have been proposed in the literature (see for instance[4] for a state of the art).

When only counterweights are used, the condition for static balancing is that the centre of mass of all moving links remains stationary for any motion of the mechanism. In this case, the forces applied by the mechanism on its base — also referred to as the shaking forces — will always be zero for any motion of the mechanism and the latter is said to be *force-balanced*. Force balancing is an important property in machine design.

In 1969, Berkof and Lowen[1] showed that it is possible to statically balance planar four-bar mechanisms without using friction or springs. They derived a set

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of conditions written in terms of the static parameters which, when satisfied, ensure that the mechanism is statically balanced, i.e., the centre of mass of the mechanism is fixed for any configuration. Later, Gosselin[3] showed that these so-called Berkof and Lowen constraints are sufficient but not necessary conditions. In the latter reference, an example of statically balanced four-bar linkage that does not satisfy the Berkof and Lowen conditions is given but no conclusion can be drawn on the existence of other such mechanisms.

In this paper, a new systematic approach is presented to determine the complete set of statically balanced (force-balanced) four-bar mechanisms. In section 2, we introduce Laurent polynomials, Newton polytopes and the Minkowski sum and we show how they can be used to find all possible factorizations of polynomials. In section 3, we formulate the kinematic constraints and the static balancing constraints as algebraic equations over real and complex variables. This leads to the problem of factorization of Laurent polynomials which can be solved using Newton polytopes and Minkowski sums. Finally, the set of necessary and sufficient conditions for statically balanced four-bar mechanisms are given.

2 Convex polytopes

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}^n$. We will write a monomial in the following form:

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} \tag{1}$$

Definition 1 *A* **Laurent polynomial** *g over a field K is a polynomial expressed in terms of the variables $x_1, ..., x_n$, where the exponents of these variables are in $\mathbb{Z}$, that is the exponents can be negative integers. They form the ring of Laurent polynomials $K[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$.*

Definition 2 *The* **support** *of a Laurent polynomial g, Supp(g), is the set of all the monomial exponents $\alpha \in \mathbb{Z}^n$ of g with non-zero coefficients.*

Example 1 *Let*

$$g = a_1 + a_2 z_1 + a_3 z_2 + a_4 z_1^{-1} z_2 + a_5 z_1^{-1} + a_6 z_2^{-1} + a_7 z_1 z_2^{-1}$$

a Laurent polynomial. The support of f is

$$\{(0,0), (1,0), (0,1), (-1,1), (-1,0), (0,-1), (1,-1)\} \tag{2}$$

Definition 3 *A set $S \subset K^n$ is* **convex** *if for any two points $x, y \in A$, the segment joining x and y is contained in A, thus if*

$$(1-\lambda)x + \lambda y \in A \quad for\ all\ x, y \in A,\ 0 \le \lambda \le 1 \tag{3}$$

Definition 4 *Let g be a Laurent polynomial in n variables. The* **Newton polytope** *$\Pi(g) \subset \mathbb{R}^n$ is defined as the convex hull of Supp(g). In other words, it is the smallest convex polytope that includes all points of the support of g. If $n = 2$, a Newton polytope is also called a* **Newton polygon**.

Example 2 *The Newton polygon of g (Example 1) is given in Figure 1.*

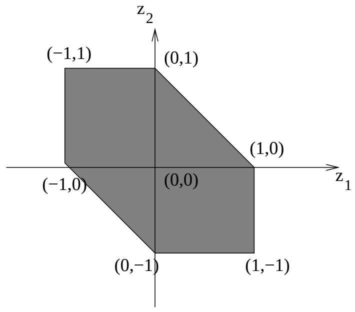


Figure 1: Newton polygon of g

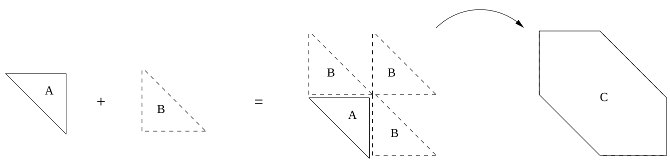


Figure 2: Minkowski sum.

We define the Minkowski sum in terms of convex sets, since every Newton polygon is in fact a convex set.

Definition 5 *The* **Minkowski sum** *of two convex sets A and $B \subset \mathbb{R}^n$ is defined as:*

$$A + B = \{a + b \mid a \in A \wedge b \in B\} \tag{4}$$

Note that $A + B$ is also a convex set.

Example 3 *Let*

$$\begin{aligned} f_1 &= a_1 z_1^{-1} + a_2 z_2^{-1} + a_3 \\ f_2 &= b_1 z_1 + b_2 z_2 + b_3 \end{aligned}$$

Let $A = \Pi(f_1)$ and $B = \Pi(f_2)$ be the Newton polygons of f_1 and f_2. The Minkowski sum $A+B$ can be computed geometrically by moving B on the boundary of A and taking the convex hull as shown in Figure 2.

Theorem 1 *Let f, g, two Laurent polynomials, then:*

$$\Pi(fg) = \Pi(f) + \Pi(g) \tag{5}$$

We refer to Ostrowski[5, 6] for a proof.

Let g be a Laurent polynomial. Using Theorem 1 it is possible to derive necessary conditions for the decomposition of g as the product of several components by looking only at the Newton polytope of f. Here is an example that will be useful in Section 4.

Example 4 *Let g be the polynomial defined in Example 1. To find all possible factorizations of g, it suffices to look at the decompositions of its Newton polygon into a Minkoski sum of Newton polygons. The possible decompositions are*

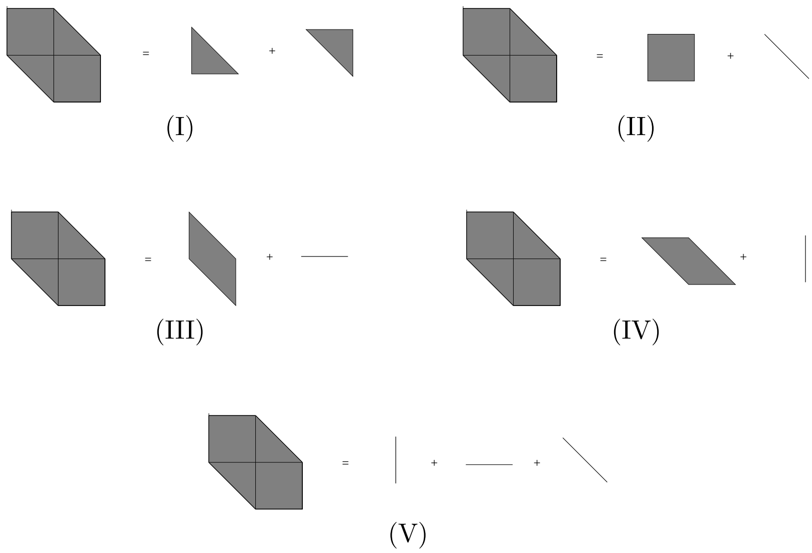


Table 1: Possible factorizations based on Newton polytopes and Minkowski sums.

given in Table 1. These components, defined by Newton polygons, can be translated back to Laurent polynomials since every integral point (point with integer coordinates) in the Newton polygon corresponds to a monomial. Therefore, we have the following decompositions of g:

$$
\begin{array}{rl}
I: & (u_1 + u_2 z_1 + u_3 z_2)(v_1 + v_2 z_1^{-1} + v_3 z_2^{-1}) \\
II: & (u_1 + u_2 z_1 + u_3 z_1 z_2^{-1} + u_4 z_2^{-1})(v_1 + v_2 z_1^{-1} z_2) \\
III: & (u_1 + u_2 z_2 + u_3 z_1 + u_4 z_1 z_2^{-1})(v_1 + v_2 z_1^{-1}) \\
IV: & (u_1 + u_2 z_1^{-1} z_2 + u_3 z_2 + u_4 z_1)(v_1 + v_2 z_2^{-1}) \\
V: & (u_1 + u_2 z_2^{-1})(v_1 + v_2 z_1)(w_1 + w_2 z_1^{-1} z_2)
\end{array}
$$

with u_i, v_i and w_i being unknown coefficients depending on the a_i ($1 \leq i \leq 7$) For more details on convexity, Newton polygons and Minkowski sums, the reader is referred to Schneider[7].

3 Planar four-bar mechanisms

3.1 Static balancing (force balancing) problem

Formulation

A planar four-bar mechanism is shown in Figure 3. It consists of four links: the base which is fixed and three links of length l_1, l_2 and l_3. The links are connected by revolute joints. The orientation of the links with respect to the fixed base is given respectively by the time variables $\theta_1(t), \theta_2(t)$ and $\theta_3(t)$. Since the mechanism has only one degree of freedom, there is a relationship between these angles which will be described below. The mass properties of the base have no influence on the equations since the base is fixed. For each of the three

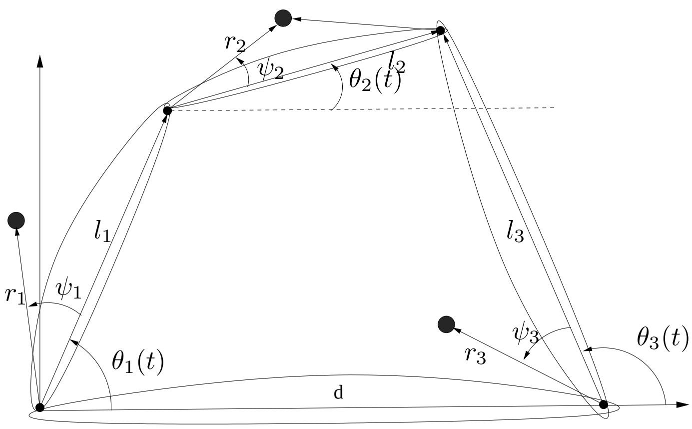


Figure 3: Four-bar mechanism.

	Type	parameters
Kinematic	Position	l_1, l_2, l_3, d
Static	Mass	m_1, m_2, m_3
	Centre of mass	$r_1, \psi_1, r_2, \psi_2, r_3, \psi_3$

Table 2: Parameters defining the four-bar mechanisms.

moving links, we can represent the mass properties of each link by a point mass m_i located at the centre of mass of the link, whose position is defined by r_i and ψ_i. Therefore, the architecture of a planar four-bar mechanism is defined by the 13 parameters given in Table 2.

It is pointed out that the second moments of inertia are omitted here since we are only interested in the statics and not the dynamics of the mechanism. The lengths (l_i), the mass (m_i) and the coordinate of the centre of mass (r_i) are non-negative real numbers. The constant angles (ψ_i) used to locate the centres of mass are in $[0, 2\pi[$.

Before stating the problem, let us compute the expression of the position vector of the global centre of mass in terms of the time variables and parameters:

$$c = \frac{1}{M}\left[m_1 r_1 e^{i\theta_1} e^{i\psi_1} + m_2\left(l_1 e^{i\theta_1} + r_2 e^{i\theta_2} e^{i\psi_2}\right) + m_3\left(d + r_3 e^{i\theta_3} e^{i\psi_3}\right)\right] \quad (6)$$

In the above expression, $c \in \mathbb{C}$ since we use a complex representation to express the x and y components of the position of the centre of mass. The real part corresponds to the x component and the imaginary part corresponds to the y component. The expression of the position of the centre of mass given by equation (6) is written in terms of the joint angles $\theta_1(t), \theta_2(t), \theta_3(t)$. Since the planar four-bar mechanism has only one degree of freedom, there exists geometric constraints between the joint angles. The problem consists in finding all possible values of the parameters such that the centre of mass expressed in equation (6) remains constant for any trajectory of the four-bar mechanism.

Eliminating θ_2

Assume l_2 is strictly positive. The case when $l_2 = 0$ is a degenerated case and will be considered in Section 4.1. Using the following closure constraint:

$$l_1 e^{i\theta_1} + l_2 e^{i\theta_2} \quad = \quad d + l_3 e^{i\theta_3} \tag{7}$$

θ_2 can easily be written in terms of θ_1 and θ_3:

$$e^{i\theta_2} = G_1 e^{i\theta_1} + G_2 e^{i\theta_3} + G_3 \tag{8}$$

where

$$G_1 = \frac{-l_1}{l_2},\ G_2 = \frac{l_3}{l_2},\ G_3 = \frac{d}{l_2}. \tag{9}$$

Moreover, since θ_2 is an angle, the value $e^{i\theta_2}$ should lie on the unit circle. Using the expression $e^{i\theta_2}$ given in equation (8), this constraint can be formulated as:

$$g(\theta_1, \theta_3) \quad = \quad 0 \tag{10}$$

where $g(\theta_1, \theta_3)$ is defined as:

$$\begin{aligned} g(\theta_1, \theta_3) \quad &:= \quad \mid e^{i\theta_2} \mid -1 & (11) \\ &= \quad \left(G_1 e^{i\theta_1} + G_2 e^{i\theta_3} + G_3\right) \left(\overline{G_1} e^{-i\theta_1} + \overline{G_2} e^{-i\theta_3} + \overline{G_3}\right) - 1. & (12) \end{aligned}$$

Since G_1, G_2 and G_3 are real, we have $\overline{G_1} = G_1, \overline{G_2} = G_2, \overline{G_3} = G_3$. Therefore $g(\theta_1, \theta_3)$ can be rewritten as:

$$g(\theta_1, \theta_3) \quad = \quad \left(G_1 e^{i\theta_1} + G_2 e^{i\theta_3} + G_3\right) \left(G_1 e^{-i\theta_1} + G_2 e^{-i\theta_3} + G_3\right) - 1. \tag{13}$$

Substituting $e^{i\theta_2}$ given by equation (8) in the expression for the centre of mass given by equation (6) yields:

$$c = \frac{1}{M} \left[F_1 e^{i\theta_1} + F_2 e^{i\theta_3} + F_3\right] \tag{14}$$

where $F_1, F_2, F_3 \in \mathbb{C}$ and defined as:

$$\begin{aligned} F_1 \quad &= \quad m_1 r_1 e^{i\psi_1} + m_2 l_1 + G_1 m_2 r_2 e^{i\psi_2} & (15) \\ F_2 \quad &= \quad m_3 r_3 e^{i\psi_3} + G_2 m_2 r_2 e^{i\psi_2} & (16) \\ F_3 \quad &= \quad m_3 d + G_3 m_2 r_2 e^{i\psi_2}. & (17) \end{aligned}$$

Since we want the centre of mass to be fixed, we want this expression to be constant. F_3 is already constant for a given mechanism since it does not depend on the time variables. Let $A = cM - F_3$ be a constant, the condition for the centre of mass to be fixed can be rewritten as:

$$f(\theta_1, \theta_3) := F_1 e^{i\theta_1} + F_2 e^{i\theta_3} \quad = \quad A. \tag{18}$$

Complex variable formulation

Now we introduce two new complex variables $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_3}$ such that $|z_1| = 1$ and $|z_2| = 1$. Recall that for any $z \in \mathbb{C}$ on the unit circle($|z| = 1$), we have $z^{-1} = \overline{z}$. Introducing these new variables z_1, z_2 in the expression defining the geometric constraint (equation (13)) gives:

$$
\begin{aligned}
g(z_1, z_2) &= (G_1 z_1 + G_2 z_2 + G_3)\left(G_1 z_1^{-1} + G_2 z_2^{-1} + G_3\right) - 1 &(19)\\
&= G_1 G_2 z_1 z_2^{-1} + G_2 G_1 z_1^{-1} z_2 + G_1 G_3 z_1 + G_3 G_1 z_1^{-1} &(20)\\
&\quad + G_2 G_3 z_2 + G_3 G_2 z_2^{-1} + (G_1 G_1 + G_2 G_2 + G_3 G_3 - 1) &(21)
\end{aligned}
$$

where $g(z_1, z_2)$ is a Laurent polynomial. From equation (14), we obtain:

$$
\begin{aligned}
f_1(z_1, z_2) &= F_1 z_1 + F_2 z_2 - A = 0 &(22)\\
f_2(z_1, z_2) &= \overline{F_1} z_1^{-1} + \overline{F_2} z_2^{-1} - \overline{A} = 0 &(23)
\end{aligned}
$$

since

$$
F_1 z_1 + F_2 z_2 - A = 0 \Rightarrow \overline{F_1}\,\overline{z_1} + \overline{F_2}\,\overline{z_2} - \overline{A} = \overline{F_1} z_1^{-1} + \overline{F_2} z_2^{-1} - \overline{A} = 0 \qquad (24)
$$

Theorem 2 *Let g be an irreducible Laurent polynomial. Let f be a Laurent polynomial(not necessarily irreducible). Let $G \subseteq \mathbb{C}^{*2}$ such that g has infinitely many zeros in G. The following are equivalent:*

1. $\forall (z_1, z_2) \in G, g(z_1, z_2) = 0 \Rightarrow f(z_1, z_2) = 0$
2. $\exists$ *Laurent polynomial $k(z_1, z_2)$ such that $f = g \cdot k$*

Proof 1 *2) $\Longrightarrow$ 1): is straightforward since if there exists k such that $f = g \cdot k$ and $g(z_1, z_2) = 0$, then $f(z_1, z_2) = 0$.*

1) $\Longrightarrow$ 2): Assume indirectly that f is not a multiple of g in the ring of Laurent polynomials. Using Bernshtein theorem[2], it follows that the number of common zeros in $\mathbb{C}^{*2}$ is at most equal to the normed mixed volume of $\Pi(f)$ and $\Pi(g)$. In particular, there are at most finitely many common zeroes. But since g has infinitely many zeros in G, this is a contradiction.

Assume that we have a non-constant trajectory given in terms of $\theta_1(t)$ and $\theta_3(t)$ such that the centre of mass is fixed. In other words, we have a set $K = \{(z_1, z_2) \in \mathbb{C}^2 \mid g(z_1, z_2) = 0\}$ which represents all the points on the trajectory and therefore contains infinitely many elements. For this set K, we want the mechanism to be statically balanced. Therefore, we want:

$$
\forall_{(z_1,z_2)\in K}\, g(z_1, z_2) = 0 \Rightarrow (f_1(z_1, z_2) = 0 \wedge f_2(z_1, z_2) = 0) \qquad (25)
$$

which can be rewritten as:

$$
\begin{aligned}
\forall_{(z_1,z_2)\in K}\, g(z_1, z_2) = 0 &\Rightarrow f_1(z_1, z_2) = 0 &(26)\\
\forall_{(z_1,z_2)\in K}\, g(z_1, z_2) = 0 &\Rightarrow f_2(z_1, z_2) = 0 &(27)
\end{aligned}
$$

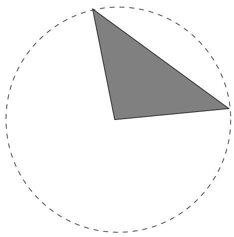


Figure 4: Degenerated case.

4 Classification of four-bar mechanisms

In this section, we derive necessary and sufficient conditions for the static balancing of planar four-bar mechanisms using the new formulation presented above. In order to use Theorem 2, g has to be irreducible, which is not necessarily the case. Therefore, we use case distinctions based on the irreducibility of g. Moreover, we want to consider the degenerated case for which at least one kinematic parameter is 0. Therefore, we can split the problem in three distinct cases:

1. **Degenerated case**: At least one kinematic parameter is 0.
2. **Irreducible case**: The kinematic parameters are all strictly positive and g is irreducible.
3. **Reducible case**: The kinematic parameters are all strictly positive and g is reducible.

4.1 Degenerated case

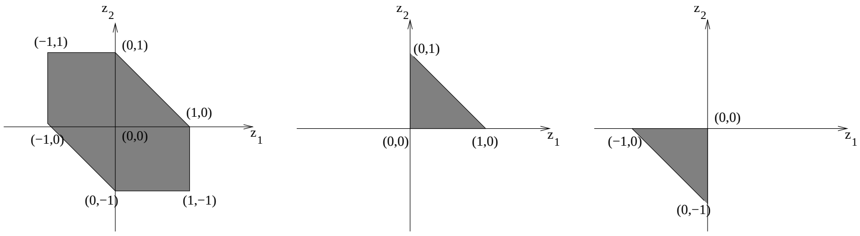
If one of the l_i (i=1,2,3) is zero and $d \neq 0$, clearly the mechanism cannot move since it has no degree of freedom. Therefore, the only case to consider is the case when $d = 0$. If all other lengths are non-zero (i.e. $l_1 \neq 0$, $l_2 \neq 0$, $l_3 \neq 0$), we obtain a triangle rotating around the origin (see Figure 4). Clearly, this mechanism is statically balanced if and only if the centre of mass of the mechanism is at the origin. The same conditions are obtained if $d = 0$ and one of the l_i is also equal to 0. In this case, the mechanism is a pendulum.

4.2 Irreducible case

Assume g is irreducible and the kinematic parameters are all strictly positive. The Newton polygon corresponding to g, f_1 and f_2 are shown in Figure 4.2.

Since the kinematic parameters cannot be zero, G_1, G_2 and G_3 are also different from zero. Therefore, the coefficients of all monomials of g are also non-zero and the Newton polygon of g cannot be smaller. However, we do not have such constraints on f_1 and f_2 since the coefficients F_1, F_2, F_3 could be equal to 0 (i.e. the Newton polygon $\Pi(f_i)$ for $i = 1, 2$ could be smaller).

Using Theorem 2, the four-bar mechanism is statically balanced if and only if there exist Laurent polynomials k_1, k_2 such that the following two conditions



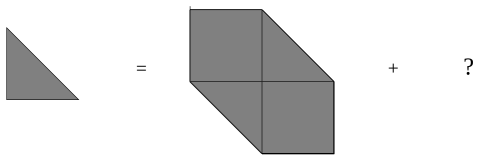
$\Pi(g)$ $\Pi(f_1)$ $\Pi(f_2)$

Figure 5: Newton polygons.

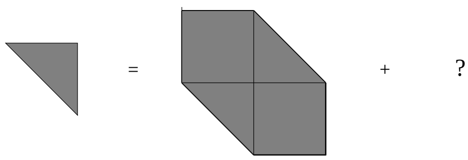
are fulfilled:

$$\begin{aligned} f_1 &= gk_1 \\ f_2 &= gk_2. \end{aligned}$$

Therefore, we can use Theorem 1 to study the Newton polygon representation of this product as a Minkowski sum:



$$\Pi(f_1) = \Pi(g) + \Pi(k_1)$$



$$\Pi(f_2) = \Pi(g) + \Pi(k_2)$$

Clearly, such non-zero polynomials k_1 and k_2 do not exist. Therefore, the only possibility is to have k_1 and k_2 be the zero polynomial. Therefore, $f_1 = f_2 = 0$ and:

$$F_1 = m_1 r_1 e^{i\psi_1} + m_2 l_1 + G_1 m_2 r_2 e^{i\psi_2} = 0 \quad (28)$$

$$F_2 = m_3 r_3 e^{i\psi_3} + G_2 m_2 r_2 e^{i\psi_2} = 0. \quad (29)$$

These conditions correspond to the conditions derived by Berkof and Lowen[1]. When g is irreducible, these conditions are sufficient and necessary.

4.3 Reducible case

Assume that g is reducible and that all kinematic parameters are strictly positive. All possible factorizations of g into irreducible components were derived in Example 4 and are shown in Table 1.

4.3.1 Reducible: Case I

Consider the first reducible case and let

$$g = h_1 h_2 \tag{30}$$

where

$$\begin{aligned} h_1 &= u_1 + u_2 z_1 + u_3 z_2 \\ h_2 &= v_1 + v_2 z_1^{-1} + v_3 z_2^{-1} \end{aligned}$$

with h_1 and h_2 irreducible polynomials and $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$ unknowns. We want to determine the values of these unknowns in terms of G_1, G_2 and G_3. Comparing coefficient in equation (30), we obtain the following system of equations:

$$\begin{aligned} \left[z_1 z_2^{-1}\right] : G_1 G_2 &= u_2 v_3 \\ \left[z_1^{-1} z_2\right] : G_1 G_2 &= u_3 v_2 \\ \left[z_1\right] : G_1 G_3 &= u_2 v_1 \\ \left[z_1^{-1}\right] : G_1 G_3 &= u_1 v_2 \\ \left[z_2\right] : G_2 G_3 &= u_3 v_1 \\ \left[z_2^{-1}\right] : G_2 G_3 &= u_1 v_3 \\ [1] : G_1 G_1 + G_2 G_2 + G_3 G_3 - 1 &= u_1 v_1 + u_2 v_2 + u_3 v_3. \end{aligned}$$

This system has no solution. Therefore, this reducible case is not physically possible.

4.3.2 Reducible: Case II

Let

$$g = h_1 h_2 \tag{31}$$

with

$$\begin{aligned} h_1 &= u_1 + u_2 z_1 + u_3 z_1 z_2^{-1} + u_4 z_2^{-1} \\ h_2 &= v_1 + v_2 z_1^{-1} z_2 \end{aligned}$$

By coefficient comparison, we obtain:

$$\begin{aligned} \left[z_1 z_2^{-1}\right] : G_1 G_2 &= u_3 v_1 \\ \left[z_1^{-1} z_2\right] : G_2 G_1 &= u_1 v_2 \\ \left[z_1\right] : G_1 G_3 &= u_2 v_1 \\ \left[z_1^{-1}\right] : G_3 G_1 &= u_4 v_2 \\ \left[z_2\right] : G_2 G_3 &= u_2 v_2 \\ \left[z_2^{-1}\right] : G_3 G_2 &= u_4 v_1 \\ [1] : G_1 G_1 + G_2 G_2 + G_3 G_3 - 1 &= u_1 v_1 + u_3 v_2 \end{aligned}$$

Solving this system of equations gives the following two constraints:

$$G_1^2 = G_2^2 \tag{32}$$

$$G_3^2 = 1 \tag{33}$$

By replacing the value of the G_i given by equation (9), these conditions can be translated into:

$$\left(\left(\frac{-l_1}{l_2}\right)^2 = \left(\frac{l_3}{l_2}\right)^2\right) \wedge \left(\left(\frac{d}{l_2}\right)^2 = 1\right). \tag{34}$$

Since all kinematic parameters are strictly positive, we obtain the following conditions:

$$l_1 = l_3 \tag{35}$$
$$l_2 = d. \tag{36}$$

When these geometric constraints are fulfilled, the equation g splits into two factors h_1 and h_2, each factor corresponding to a kinematic mode. We need to investigate these two kinematic modes separately since it might be possible to find static balancing conditions for one mode which are not valid for the other mode, or *vice versa*.

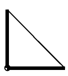
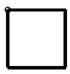
$\mathbf{h_1 = 0}$: For a mechanism in this mode to be statically balanced, the following conditions should be fulfilled (see equation(26, 27)):

$$\forall_{(z_1,z_2)\in K} h_1(z_1,z_2) = 0 \quad \Rightarrow \quad f_1(z_1,z_2) = 0$$
$$\forall_{(z_1,z_2)\in K} h_1(z_1,z_2) = 0 \quad \Rightarrow \quad f_2(z_1,z_2) = 0.$$

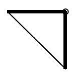
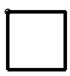
Since we assumed h_1 is irreducible, from Theorem 2, there must exist Laurent polynomials k_1 and k_2 such that:

$$f_1 = h_1 k_1 \tag{37}$$
$$f_2 = h_1 k_2. \tag{38}$$

Looking at the corresponding Newton polygons:

 =  + ?

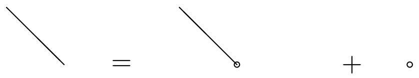
$$\Pi(f_1) = \Pi(h_1) + \Pi(k_1)$$

 =  + ?

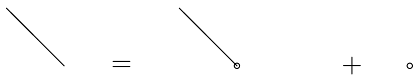
$$\Pi(f_2) = \Pi(h_1) + \Pi(k_2)$$

the only solution again is k_1 and k_2 being zero Laurent polynomials, meaning that $f_1 = f_2 = 0$ and we obtain the same conditions($F_1 = F_2 = 0$) as in the irreducible case.

$\mathbf{h_2 = 0}$: The second mode is more interesting. First we obtain the trivial solution $f_1 = f_2 = 0$ as in every cases ($F_1 = F_2 = 0$). But we obtain also a second solution. Using the same approach as before, we see this time that it is possible to find non-zero k_1 and k_2 (actually constant Laurent polynomials). We obtain the following decomposition:



$$\Pi(f_1) \quad = \quad \Pi(h_2) \quad + \quad \Pi(k_1)$$



$$\Pi(f_2) \quad = \quad \Pi(h_2) \quad + \quad \Pi(k_2)$$

where the coefficients F_1 and F_2 are non zero. Actually, the Newton polygons of f_1 and f_2 above take into account the fact that F_3 (the constant term) is zero. Hence, we know that it is possible to find a statically balanced mechanism in this mode. The expression for h_2 is:

$$\begin{aligned} h_2 &= v_1 + v_2 z_1^{-1} z_2 \\ &= G_1 G_3 + G_2 G_3 z_1^{-1} z_2 \\ &= \frac{-l_1}{l_2}\frac{d}{l_2} + \frac{l_3}{l_2}\frac{d}{l_2} z_1^{-1} z_2 \\ &= \frac{-l_1}{l_2} + \frac{l_1}{l_2} z_1^{-1} z_2 \\ &= \frac{l_1}{l_2}(-1 + z_1^{-1} z_2). \end{aligned}$$

Therefore, $h_2 = 0$ if and only if $z_1 = z_2$ or in other words $\theta_1 = \theta_3$ which corresponds to the kinematic mode shown in Figure 6.

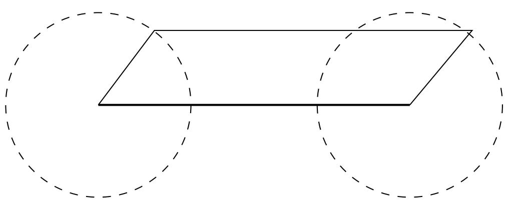


Figure 6: Kinematic mode h_2.

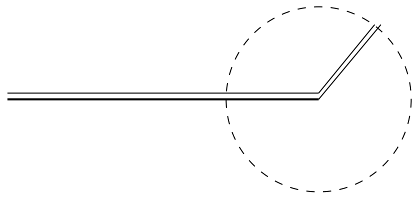


Figure 7: Kinematic mode h_2.

Since $l_1 = l_3$ and $\theta_1 = \theta_3$, f can be rewritten as:

$$\begin{aligned} f(z_1, z_2) &= F_1 z_1 + F_2 z_2 \\ &= (F_1 + F_2) z_1 \\ &= (m_1 r_1 e^{i\psi_1} + m_2 l_1 + \frac{-l_1}{l_2} m_2 r_2 e^{i\psi_2} + m_3 r_3 e^{i\psi_3} + \frac{l_3}{l_2} m_2 r_2 e^{i\psi_2}) z_1 \\ &= (m_1 r_1 e^{i\psi_1} + m_2 l_1 + m_3 r_3 e^{i\psi_3}) z_1 \end{aligned}$$

f is constant if and only if the coefficient $F_1 + F_2 = m_1 r_1 e^{i\psi_1} + m_2 l_1 + m_3 r_3 e^{i\psi_3}$ is 0 which gives a sufficient and necessary statically balancing constraint for this kinematic mode. This is the solution that was found by Gosselin[3].

4.3.3 Reducible: Case III

Let

$$g = h_1 h_2 \tag{39}$$

where

$$\begin{aligned} h_1 &= u_1 + u_2 z_2 + u_3 z_1 + u_4 z_1 z_2^{-1} \\ h_2 &= v_1 + v_2 z_1^{-1}. \end{aligned}$$

Using coefficient comparison, we obtain the following conditions on the kinematic parameters:

$$l_1 = d \tag{40}$$

$$l_2 = l_3. \tag{41}$$

For the kinematic mode corresponding to $h_1 = 0$, the mechanism can be statically balanced only if $F_1 = F_2 = 0$. However, we can find less restrictive balancing constraints for the $h_2 = 0$ kinematic mode. It can be shown that $v_1 = G_2 G_3$ and $v_2 = G_2 G_1$, therefore

$$h_2 = G_2 G_3 + G_1 G_2 z_1^{-1} = \frac{l_1}{l_2} \left(1 - z_1^{-1}\right). \tag{42}$$

In other words, $h_2 = 0$ if and only if $z_1 = e^{i\theta_1} = 1$ which implies that $\theta_1 = 0$. This kinematic mode is shown in Figure 7.

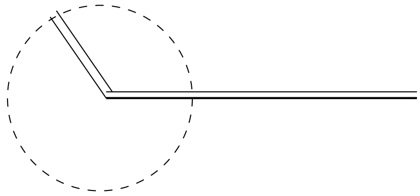


Figure 8: Kinematic mode h_2.

In this mode, we have

$$F_1 e^{i\theta_1} + F_2 e^{i\theta_2} = F_1 + F_2 e^{i\theta_2} = constant \Leftrightarrow F_2 = 0. \quad (43)$$

Therefore, we obtain the following unique constraint for the static balancing of the mechanism when $\theta_1 = 0$:

$$F_2 = m_3 r_3 e^{i\psi_3} + G_2 m_2 r_2 e^{i\psi_2} = m_3 r_3 e^{i\psi_3} + m_2 r_2 e^{i\psi_2} = 0. \quad (44)$$

4.3.4 Reducible: Case IV

This case is symmetric to case III. This reducible case corresponds to the following conditions on the kinematic parameters:

$$l_3 = d \quad (45)$$
$$l_1 = l_2. \quad (46)$$

We obtain $\theta_3 = \pi$ for the kinematic mode related to h_2 (see Figure 8).

In this mode, we have

$$F_1 e^{i\theta_1} + F_2 e^{i\theta_2} = F_1 e^{i\theta_1} - F_2 = constant \Leftrightarrow F_1 = 0. \quad (47)$$

Therefore, we obtain the constraint:

$$F_1 = m_1 r_1 e^{i\psi_1} + m_2 l_1 + G_1 m_2 r_2 e^{i\psi_2} = m_1 r_1 e^{i\psi_1} + m_2 l_1 - m_2 r_2 e^{i\psi_2} = 0. \quad (48)$$

4.3.5 Reducible: Case V

In this case, we obtain:

$$l_1 = l_2 = l_3 = d \quad (49)$$

We get 3 possible modes which corresponds to case 2,3 and 4 mentioned above. For case 2, $l_1 = l_3$ and $d = l_2$. These constraints are obviously also valid in case V. The same holds for case 3 and 4. Therefore, the 3 kinematic modes of case V are special cases of case II, III and IV.

4.4 Summary

All necessary and sufficient conditions for the static balancing of four-bar mechanisms are summarized in Table 3.

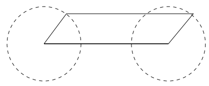
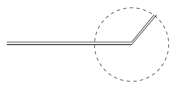
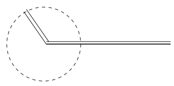
Geometric constraint	Kinematic mode	Static constraints
$l_1 = l_3$ $l_2 = d$	$\theta_1 = \theta_3$ 	$F_1 + F_2 = 0$
$l_1 = d$ $l_2 = l_3$	$\theta_1 = 0$ 	$F_2 = 0$
$l_1 = l_2$ $l_3 = d$	$\theta_3 = \pi$ 	$F_1 = 0$
All other cases		$F_1 = F_2 = 0$

Table 3: Sufficient and necessary conditions for statically balanced planar four-bar mechanisms

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