Computing difference-differential Gröbner bases and difference-differential dimension polynomials *

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Abstract
Difference-differential Gröbner bases and the algorithms were introduced by M.Zhou and F.Winkler (2006). In this paper we will make further investigations for the key concept of S-polynomials in the algorithm and we will improve technically the algorithm. Then we apply the algorithm to compute the difference-differential dimension polynomial of a difference-differential module and of a system of linear partial difference-differential equations. Also, in cyclic module case, we present an algorithm to compute the difference-differential dimension polynomials in two variables with the Gröbner basis.

Keywords: difference-differential Gröbner basis, S-polynomial, difference-differential dimension polynomial.

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1 Introduction

The notion of Gröbner basis, being a powerful tool to solve various problems by algorithmic way in polynomial ideal theory, have been explored in differential algebra and difference-differential algebra by many authors. Although the attempt to imitate Gröbner basis method in the context of differential ideals of a ring of differential polynomials has been unsuccessful to date, the theory of

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Gröbner bases in free modules over various rings of differential (or difference-differential) operators has been developed, see Noumi (1988), Takayama (1989), Oaku and Shimoyama(1994), Carra Ferro(1997), Insa and Pauer(1998), Pauer and Unterkircher (1999), Levin(2000), Zhou and Winkler(2006). This theory is essential for many problems of linear difference-differential equations such as the dimension of the space of solutions and the computation of difference-differential dimension polynomials.

Difference-differential Gröbner bases and the algorithms were introduced by Zhou and Winkler(2006). It is based on generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$. In the algorithms for computation of the Gröbner bases, the S-polynomial of f and g with respect to $\Lambda_j E$ and v

$$S(j, f, g, v) = \frac{v}{lt_j(f)} \frac{f}{lc_j(f)} - \frac{v}{lt_j(g)} \frac{g}{lc_j(g)}$$

play a important role. Where $\Lambda_j E$ is an orthant and v is a generator of R-module $V(j, f, g)$. In the algorithm we have to compute S-polynomials for every orthant and every generator of $V(j, f, g)$. So we have to compute a set of finite generators of $V(j, f, g)$. In this paper we give a technical improving of the algorithms for computation of difference-differential Gröbner bases such that in some case computing the set of finite generators of $V(j, f, g)$ is unnecessary and S-polynomials just involves every orthant.

The concept of the differential dimension polynomial was introduced in Kolchin(1964) as a dimensional description of some differential field extension. Johnson(1974) proved that the differential dimension polynomial of a differential field extension coincides with the Hilbert polynomial of some filtered differential module. This result allowed to compute differential dimension polynomials using the Gröbner basis technique and other basis method. Since then various problems of differential algebra involving differential dimension polynomials were studied. See Levin and Mikhalev(1987), Kondrateva *et al*(1999). The concepts of the difference dimension polynomial and the difference-differential dimension polynomial were introduced first in Levin(1978), Mikhalev and Pankratev (1989) respectively. They play the same role in difference algebra (resp. difference-differential algebra) as Hilbert polynomials in commutative algebra or differential dimension polynomials in differential algebra. The notion of difference-differential dimension polynomial can be used for the study of dimension theory of difference-differential field extension as well as systems of algebraic difference-differential equations.

In Mikhalev and Pankratev (1989) the authors proved existence of difference-differential dimension polynomials $\phi(t)$ associated with M by classical Gröbner basis methods of computation of Hilbert polynomials. The proof is based on the fact that the ring of difference-differential operators over the difference-differential field R is isomorphic to the factor ring of the ring of generalized polynomials $R[x_1, \cdots, x_{m+2n}]$ (where $x_i a = a x_i + \delta_i(a)$ $(1 \leq i \leq m)$, $x_{m+j} a = \alpha_j(a) x_{m+j}$ and $x_{m+n+j} a = \alpha_j^{-1}(a) x_{m+n+j}$ $(1 \leq j \leq n)$ for any $a \in R$) by the ideal I generated by the polynomials $x_{m+j} x_{m+n+j} - 1$ $(1 \leq j \leq n)$. However, a similar approach to difference-differential dimension polynomials in two variables (one for difference part and another for differential part) is unsuccessful. Levin(2000) investigated the difference-differential dimension polynomials in two variables with characteristic set method. The method of Levin is rather delicate but no general algorithm for computing the characteristic set, although for cyclic modules an algorithm was given.

Difference-differential Gröbner bases give a new approach for algorithmic computing the difference-differential dimension polynomials in one or two variables. In this paper we present a new algorithmic approach to compute the difference-differential dimension polynomials $\phi(t)$ associated with M by the difference-differential Gröbner basis method. This method is more direct and simpler than the traditional Gröbner basis method. Also, an new algorithmic approach based on difference-differential Gröbner basis has presented to compute the difference-differential dimension polynomials in two variables for cyclic modules. This paper is organized as following: Section 2 is preliminaries. In section 3 the improving of the algorithm for computing difference-differential Gröbner bases is described. In section 4 we study the difference-differential dimension polynomials in one variable using difference-differential Gröbner basis method. Section 5 is contributed to the difference-differential dimension polynomials in two variables for which the traditional Gröbner basis method does not work.

2 Preliminaries

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_-$ and $\mathbb{Q}$ denotes the sets of all integers, all nonnegative integers, all nonpositive integers, and all rational numbers, respectively. By a ring we always mean an associative ring with a unit. By the module over a ring A we mean a unitary left A-module.

Let R be a commutative noetherian ring, $\Delta = \{\delta_1, \cdots, \delta_m\}$ and $\Sigma = \{\sigma_1, \cdots, \sigma_n\}$ be set of derivations and automorphisms of the ring R, respectively, such that $\beta(x) \in R$ and $\beta(\gamma(x)) = \gamma(\beta(x))$ hold for any $\beta, \gamma \in \Delta \bigcup \Sigma$ and $x \in R$. Then R is called a difference-differential ring with the basic set of derivations Δ and the basic set of automorphisms Σ, or shortly a Δ-Σ-ring. If R is a field, then it is called a Δ-Σ-field.

If R is a Δ-Σ-ring, then Λ will denote the commutative semigroup of elements of the form

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \tag{2.1}$$

where $(k_1, \cdots, k_m) \in \mathbb{N}^m$ and $(l_1, \cdots, l_n) \in \mathbb{Z}^n$. This semigroup contains the free commutative semigroup Θ generated by the set Δ and free commutative semigroup Γ generated by the set Σ. The subset $\{\sigma_1, \cdots, \sigma_n, \sigma_1^{-1}, \cdots, \sigma_n^{-1}\}$ of Λ will be denoted by Σ^*.

Let R be a Δ-Σ-ring and the semigroup Λ be as above. Then an expression of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda, \tag{2.2}$$

where $a_\lambda \in R$ for all $\lambda \in \Lambda$ and only finitely many coefficients a_λ are different from zero, is called a difference-differential operator (or shortly a Δ-Σ-operator) over R. Two Δ-Σ-operators $\sum_{\lambda\in\Lambda} a_\lambda\lambda$ and $\sum_{\lambda\in\Lambda} b_\lambda\lambda$ are equal if and only if $a_\lambda = b_\lambda$ for all $\lambda \in \Lambda$.

The set of all Δ-Σ-operators over a Δ-Σ-ring R is a ring with the following fundamental relations

$$\sum_{\lambda\in\Lambda} a_\lambda\lambda + \sum_{\lambda\in\Lambda} b_\lambda\lambda = \sum_{\lambda\in\Lambda}(a_\lambda + b_\lambda)\lambda,$$

$$a(\sum_{\lambda\in\Lambda} a_\lambda\lambda) = \sum_{\lambda\in\Lambda}(aa_\lambda)\lambda,$$

$$(\sum_{\lambda\in\Lambda} a_\lambda\lambda)\mu = \sum_{\lambda\in\Lambda} a_\lambda(\lambda\mu),$$

$$\delta a = a\delta + \delta(a), \qquad \tau a = \tau(a)\tau, \qquad (2.3)$$

for all $a_\lambda, b_\lambda \in R$, $\lambda, \mu \in \Lambda$, $a \in R$, $\delta \in \Delta$, $\tau \in \Sigma^*$. Note that the elements in Δ and Σ^* do not commute with the elements in R, and then the "terms" $\lambda \in \Lambda$ do not commute with the coefficients $a_\lambda \in R$.

DEFINITION 2.1. The ring of all Δ-Σ-operators over a Δ-Σ-ring R is called the ring of difference-differential operators (or shortly the ring of Δ-Σ-operators) over R, it will be denoted by D. A left D-module M is called a difference-differential module(or a Δ-Σ-module). If M is finitely generated as a left D-module, then M is called a finitely generated Δ-Σ-module.

DEFINITION 2.2. If $\mathbb{Z}^n$ is a union of finitely many $\mathbb{Z}_j^{(n)}$:

$$\mathbb{Z}^n = \bigcup_{j=1}^{k} \mathbb{Z}_j^{(n)}$$

where $\mathbb{Z}_j^{(n)}$, $j = 1, \cdots, k$, satisfy following conditions:

(i) $(0, \cdots, 0) \in \mathbb{Z}_j^{(n)}$, and $\mathbb{Z}_j^{(n)}$ does not contain any pair of invertible elements $c = (c_1, \cdots, c_n) \neq 0$ and $c^{-1} = (-c_1, \cdots, -c_n)$,

(ii) $\mathbb{Z}_j^{(n)}$ is finitely generated sub-semigroup of $\mathbb{Z}^n$,

(iii) the group generated by $\mathbb{Z}_j^{(n)}$ is $\mathbb{Z}^n$;

then $\{\mathbb{Z}_j^{(n)},\ j = 1, \cdots, k\}$ is called an orthant decomposition of $\mathbb{Z}^n$ and $\mathbb{Z}_j^{(n)}$ is called the j-th orthant of the decomposition.

EXAMPLE 2.1. Let $\{\mathbb{Z}_1^{(n)}, \cdots, \mathbb{Z}_{2^n}^{(n)}\}$ be all distinct Cartesian products of n sets each of which is either $\mathbb{N}$ or $\mathbb{Z}_-$. Then it is an orthant decomposition of $\mathbb{Z}^n$. The set of generators of $\mathbb{Z}_j^{(n)}$ as a semigroup is

$$\{(c_1, 0, \cdots, 0), (0, c_2, 0, \cdots, 0), \cdots, (0, \cdots, 0, c_n)\},$$

where c_j is either 1 or -1, $j = 1, \cdots, n$. We call this decomposition the canonical orthant decomposition of $\mathbb{Z}^n$. $\square$

Let $\{\mathbb{Z}_j^{(n)}, \quad j=1,\cdots,k\}$ be an orthant decomposition of $\mathbb{Z}^n$. A element $a=(k_1,\cdots,k_m,l_1,\cdots,l_n)$ of $\mathbb{N}^m\times\mathbb{Z}^n$ is called in the j-th orthant, if the n-tuples $(l_1,\cdots,l_n)$ is in the j-th orthant $\mathbb{Z}_j^{(n)}$ of $\mathbb{Z}^n$.

Moreover,let λ be the form of (2.1). Then the subset Λ_j of Λ

$$\Lambda_j=\{\lambda=\delta_1^{k_1}\cdots\delta_m^{k_m}\sigma_1^{l_1}\cdots\sigma_n^{l_n} \mid (l_1,\cdots,l_n)\in\mathbb{Z}_j^{(n)}\},$$

where $\mathbb{Z}_j^{(n)}$ is the j-th orthant of the decomposition of $\mathbb{Z}^n$, is called j-th orthant of Λ.

DEFINITION 2.3. Let $\{\mathbb{Z}_j^{(n)}, \quad j=1,\cdots,k\}$ be an ortant decomposition of $\mathbb{Z}^n$. A total order $\prec$ on $\mathbb{N}^m\times\mathbb{Z}^n$ is called a generalized term order on $\mathbb{N}^m\times\mathbb{Z}^n$ with respect to the decomposition, if the following conditions hold:

(i) $(0,\cdots,0)$ is the smallest elements in $\mathbb{N}^m\times\mathbb{Z}^n$,

(ii)if $a\prec b$, then for any c such that c and b are in the same orthant, $a+c\prec b+c$. Where $a,b,c\in\mathbb{N}^m\times\mathbb{Z}^n$.

EXAMPLE 2.2. Given the canonical orthant decomposition of $\mathbb{Z}^n$, an order "$\prec'$" in $E=\{e_1,\cdots,e_q\}$, for two elements $(a,e_i)=(k_1,\cdots,k_m,l_1,\cdots,l_n,e_i)$ and $(b,e_j)=(r_1,\cdots,r_m,s_1,\cdots,s_n,e_j)$ of $\mathbb{N}^m\times\mathbb{Z}^n\times E$ define:

$$|a|=k_1+\cdots+k_m+|l_1|+\cdots+|l_n|.$$

$$(a,e_i)\prec(b,e_j)\Longleftrightarrow(|a|,e_i,k_1,\cdots,k_m,|l_1|,\cdots,|l_n|,l_1,\cdots,l_n)$$
$$<(|b|,e_j,r_1,\cdots,r_m,|s_1|,\cdots,|s_n|,s_1,\cdots,s_n) \ \textit{in lexicographic order.}$$

Then "$\prec$" is a generalized term order on $\mathbb{N}^m\times\mathbb{Z}^n\times E$.

Let Λ be the semi-group in which the elements are of the form (2.1). Since Λ is isomorphic to $\mathbb{N}^m\times\mathbb{Z}^n$ as a semigroup, a generalized term order "$\prec$" on $\mathbb{N}^m\times\mathbb{Z}^n$ would define an order on Λ. we also call it a generalized term order on Λ. The notion can be easily defined in a finitely generated free D-module F. For more details can be find in Zhou and Winkler(2006).

3 Improving the algorithm for computing difference-differential Gröbner bases

Let R be a Δ-Σ-field and D be the ring of Δ-Σ-operators over R, and let F be a finitely generated free D-module with a set of free generators $E=\{e_1,\cdots,e_q\}$. Then F can be considered as an R-vector space generated by the set of all elements of the form λe_i ($i=1,\cdots,q$, where $\lambda\in\Lambda$). This set will be denoted by ΛE and its elements will be called "terms" of F. In particular the elements of Λ will be called "terms" of D. If "$\prec$" is a generalized term order on $\mathbb{N}^m\times\mathbb{Z}^n\times B$ then "$\prec$" would define a generalized term order on ΛE.

Let $u=\lambda_1e_i$,$v=\lambda_2e_j\in\Lambda E$ and

$$\lambda_1=\delta_1^{k_1}\cdots\delta_m^{k_m}\sigma_1^{l_1}\cdots\sigma_n^{l_n}, \qquad \lambda_2=\delta_1^{r_1}\cdots\delta_m^{r_m}\sigma_1^{s_1}\cdots\sigma_n^{s_n}$$

be two elements in Λ and they are in a same orthant. If $i = j$, and $r_\nu \leq k_\nu$,$|s_\mu| \leq |l_\mu|$ for $\nu = 1, \cdots, m$, $\mu = 1, \cdots, n$, then λ_1 is called a multiple of λ_2 and u is called a multiple of v.

Let "$\prec$" be a generalized term order on ΛE, F be a finitely generated free D-module,

$$f = a_1\lambda_1 e_{j_1} + \cdots + a_d\lambda_d e_{j_d} \in F \tag{3.1}$$

Then

$$lt(f) = max_\prec\{\lambda_i e_{j_i} | i = 1, \cdots, d\}$$

is called the leading term of f. If $\lambda_i e_{j_i} = lt(f)$, then $lc(f) = a_i$ is called the leading coefficient of f.

In general $lt(\lambda f) = \lambda lt(f)$ is not true. But the following Lemma was proved in Zhou and Winkler(2006).

LEMMA 3.1. Let F be a finitely generated free D-module and $0 \neq f \in F$. Then the following assertions hold:

(i) If $\lambda \in \Lambda$, then $lt(\lambda f) = max_\prec\{\lambda \cdot u_i\}$ where u_i are terms of f and then $lt(\lambda f) = \lambda \cdot u$ for an unique term u of f.

(ii) If $lt(f) \in \Lambda_j e$ then for any $\lambda \in \Lambda_j$

$$lt(\lambda f) = \lambda \cdot lt(f) \in \Lambda_j E$$

(iii) For each j, there exists some $\lambda \in \Lambda$ and an unique term u_j of f such that

$$lt(\lambda f) = \lambda \cdot u_j \in \Lambda_j E.$$

We will write $lt_j(f)$ for this term u_j.

The following theorems are main results in Zhou and Winkler(2006) for the computation of difference-differential Gröbner bases.

THEOREM 3.1. Let F be a free D-module and $\prec$ be a generalized term order on ΛE, G be a finite subset of $F\backslash\{0\}$ and W be the submodule in F generated by G. Then G is a Gröbner basis of W if and only if for all Λ_j, for all $g_i,\ g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G.Where

$$S(j, f, g, v) = \frac{v}{lt_j(f)}\frac{f}{lc_j(f)} - \frac{v}{lt_j(g)}\frac{g}{lc_j(g)}$$

is the S-polynomial of f and g with respect to j and v.

$V(j, f, g)$ be a finite system of generators of the $R[\Lambda_j]$-module

$$_{R[\Lambda_j]}\langle lt(\lambda f) \in \Lambda_j E \mid \lambda \in \Lambda\rangle \bigcap \ _{R[\Lambda_j]}\langle lt(\eta g) \in \Lambda_j E \mid \eta \in \Lambda\rangle$$

THEOREM 3.2. Let F be a free D-module and $\prec$ be a generalized term order on ΛE, G be a finite subset of $F\backslash\{0\}$ and W be the submodule in F generated by G. For each Λ_j and $f, g \in F\backslash\{0\}$ let $V(j, f, g)$ and $S(j, f, g, v)$ be as above. Then by the following algorithm a Gröbner basis of W can be computed:

Input: $G = \{g_1, \cdots, g_\mu\}$ which is a set of generators of W
output: $G' = \{g'_1, \cdots, g'_\nu\}$ which is a Gröbner basis of W
Begin
$G_0 := G$
While there exist $f, g \in G_i$ and $v \in V(j, f, g)$ such that $S(j, f, g, v)$ reduced to $r \neq 0$ by G_i
Do $G_{i+1} := G_i \bigcup \{r\}$
If $G_{i+1} = G_i$ **then** $G_{i+1} = G'$
End

In general, the finite system of generators $V(j, f, g)$ may be have more than one elements. And we have to compute S-polynomials $S(j, g_i, g_k, v)$ for every $v \in V(j, f, g)$. The following example illustrate the case.

EXAMPLE 3.1. Let $F = D$ be a cyclic free Δ-Σ-module and the generalized term order on Λ as in Example 2.2. Let

$$f = \sigma_1^{-2}\sigma_2^3 + \sigma_1^5\sigma_2^{-2}$$

$$g = \sigma_1^2\sigma_2^3 + \sigma_1^{-1}$$

Denote the orthant $\{\sigma_1^{-a}\sigma_2^b | a, b \in \mathbb{N}\}$ by Λ_2. Then

$$lt(\sigma_1^{-2}\sigma_2^{-1}f) = lt(\sigma_1^{-4}\sigma_2^2 + \sigma_1^3\sigma_2^{-3}) = \sigma_1^{-4}\sigma_2^2 \in \Lambda_2$$

and it is a generator of the module $V_1 =_{R[\Lambda_2]} \langle lt(\lambda f) \in \Lambda_2 \mid \lambda \in \Lambda \rangle$.

Similarly,

$$lt(\sigma_2^2 f) = \sigma_1^{-2}\sigma_2^5 \in \Lambda_2$$
$$lt(\sigma_1^{-1}\sigma_2^1 f) = \sigma_1^{-3}\sigma_2^4 \in \Lambda_2$$
$$lt(\sigma_1^{-3}\sigma_2^{-2} f) = \sigma_1^{-5}\sigma_2 \in \Lambda_2$$
$$lt(\sigma_1^{-4}\sigma_2^{-3} f) = \sigma_1^{-6} \in \Lambda_2$$

are all generators of the module V_1.

Because $lt(\sigma_1^{-2}g) = \sigma_1^{-3} \in \Lambda_2$ is the only generator of the module

$$V_2 =_{R[\Lambda_2]} \langle lt(\lambda g) \in \Lambda_2 \mid \lambda \in \Lambda \rangle$$

the generator set of $V_1 \bigcap V_2$ is

$$V(j, f, g) = \{v_1 = \sigma_1^{-4}\sigma_2^2, \quad v_2 = \sigma_1^{-3}\sigma_2^4, \quad v_3 = \sigma_1^{-5}\sigma_2, \quad v_4 = \sigma_1^{-6}\}$$

Now

$$S(2, f, g, v_1) = \sigma_1^{-2}\sigma_2^{-1}f - \sigma_1^{-3}\sigma_2^2 g = \sigma_1^3\sigma_2^{-3} - \sigma_1^{-1}\sigma_2^{-1}$$

and

$$S(2, f, g, v_2) = \sigma_1^4\sigma_2^{-1} - \sigma_2 = \sigma_1\sigma_2^2 \; S(2, f, g, v_1)$$
$$S(2, f, g, v_3) = \sigma_1^2\sigma_2^{-4} - \sigma_1^{-2}\sigma_2^{-2} = \sigma_1^{-1}\sigma_2^{-1} \; S(2, f, g, v_1)$$
$$S(2, f, g, v_4) = \sigma_1\sigma_2^{-5} - \sigma_1^{-3}\sigma_2^{-3} = \sigma_1^{-2}\sigma_2^{-2} \; S(2, f, g, v_1)$$

□

The above example suggest that every $S(j, f, g, v_i)$ can be reduced to 0 by an $S(j, f, g, v)$. In fact, we have the following Proposition.

PROPOSITION 3.1. Let $f, g \in F$ and F , $\prec$, $S(j, f, g, v)$, $V(j, f, g)$ be in theorem 3.1. Choose u such that $u = lt(\lambda f) = lt(\mu g) \in \Lambda_j E$ and λ is the form σ^{t_i} , $t_i \in \mathbb{Z}^n$ (This means λ is invertible). Then for every $v \in V(j, f, g)$, the S-polynomial $S(j, f, g, v)$ can be reduced to 0 by $S(j, f, g, u)$. Then In the algorithm for computation of difference-differential Gröbner bases, if we get an above S-polynomial $S(j, f, g, u)$ and it is reduced w.r.t. G_i, then the computation of $S(j, f, g, v)$ for all $v \in V(j, f, g)$ may be eliminated.

PROOF. By Zhou and Winkler(2006) Lemma 3.3, the λ can be find, and $u = lt(\lambda f) = \lambda lt_j(f)$. Then

$$S(j, f, g, u) = \frac{u}{lt_j(f)} \frac{f}{lc_j(f)} - \frac{u}{lt_j(g)} \frac{g}{lc_j(g)}$$

Suppose

$$S(j, f, g, v) = \frac{v}{lt_j(f)} \frac{f}{lc_j(f)} - \frac{v}{lt_j(g)} \frac{g}{lc_j(g)}$$

and $v = lt(\lambda_1 f) = lt(\mu_1 g) \in \Lambda_j E$, then $v = \lambda_1 lt_j(f) = \lambda_1 \lambda^{-1} u$. This means that

$$S(j, f, g, v) = \lambda_1 \lambda^{-1} S(j, f, g, u)$$

Then $S(j, f, g, v)$ can be reduced to 0 by $S(j, f, g, u)$. □

The S-polynomial $S(j, f, g, u)$ described in Proposition 3.1 will be called an S-polynomial w.r.t. an invertible λ.

In the algorithm given by theorem 3.2, every steps when we get an S-polynomial $S(j, f, g, v)$ we have to reduce it to r by G_i. Also we have to compute S-polynomials for every $v \in V(j, f, g)$. If there is an S-polynomial $S(j, f, g, u)$ w.r.t. an invertible λ already in G_i, then all $S(j, f, g, v)$, $v \in V(j, f, g)$, would be reduced to 0 by G_i and we needn't to compute the $S(j, f, g, v)$.

EXAMPLE 3.2. Let $M = D$ be a free cyclic Δ-Σ-module.

$$f = \sigma_1 \sigma_2 (\sigma_1 + 1)$$

$$g = \sigma_1 \sigma_2 (\sigma_2 + 1)$$

Let the generalized term order $\prec$ on Λ is the same as in Example 2.2, we compute a Gröbner basis of $W = \langle f,\ g \rangle$. Note that now we have no differential operators so every S-polynomial $S(j, f, g, u)$ is invertible.

Denote the orthants as

$$\Lambda_1 = \{\sigma_1^a \sigma_2^b | a, b \in \mathbb{N}\}$$

$$\Lambda_2 = \{\sigma_1^{-a} \sigma_2^b | a, b \in \mathbb{N}\}$$

$$\Lambda_3 = \{\sigma_1^{-a} \sigma_2^{-b} | a, b \in \mathbb{N}\}$$

$$\Lambda_4 = \{\sigma_1^a \sigma_2^{-b} | a, b \in \mathbb{N}\}$$

Then

$$S(1,f,g) = {\sigma_1}^{-1}f - {\sigma_2}^{-1}g = \sigma_2 - \sigma_1 = f_1$$

Because

$${\sigma_1}^{-2}{\sigma_2}^{-1}f = {\sigma_1}^{-1} + 1$$

in which the leading term is in Λ_2 and

$${\sigma_1}^{-1}{\sigma_2}^{-1}g = \sigma_2 + 1$$

is so too, we have

$$S(2,f,g) = \sigma_2({\sigma_1}^{-1}+1) - {\sigma_1}^{-1}(\sigma_2+1) = \sigma_2 - {\sigma_1}^{-1} = f_2$$

Similarly, we have

$$S(3,f,g) = {\sigma_2}^{-1} - {\sigma_1}^{-1} = f_3$$
$$S(4,f,g) = {\sigma_2}^{-1} - \sigma_1 = f_4$$

$$S(1,f,f_1) = {\sigma_1}^{-1}{\sigma_2}^{-1}f + f_1 = \sigma_2 + 1 = g_1$$

Then $f_1 = g_1 - (\sigma_1+1)$, this means f_1 can be reduced to $\sigma_1+1 = g_2$. Similarly,

$$f_2 = g_1 - ({\sigma_1}^{-1}+1) \longrightarrow ({\sigma_1}^{-1}+1) = g_3$$

$$f_3 = -{\sigma_1}^{-1}{\sigma_2}^{-1}f_1 \longrightarrow 0$$

$$f_4 = -g_2 + ({\sigma_2}^{-1}+1) \longrightarrow ({\sigma_2}^{-1}+1) = g_4$$

Then it is easy to check

$$\{g_1, g_2, g_3, g_4\} = \{\sigma_2+1, \sigma_1+1, {\sigma_1}^{-1}+1, {\sigma_2}^{-1}+1\}$$

is a Gröbner basis of N. $\square$

4 Computing difference-differential dimension polynomials

Let R be a Δ-Σ-field, D the ring of Δ-Σ-operators over R, M a finitely generated Δ-Σ-module(i.e. a finitely generated difference-differential-module), F a finitely generated free Δ-Σ-module.

For $\lambda \in \Lambda$ of the form (2.1), let $ord\lambda = k_1 + \cdots + k_m + |l_1| + \cdots + |l_n|$. Also, for $w = \lambda e_i \in \Lambda E$ of a term of F, let $ord\ w = ord\lambda$. If $u = \sum_{\lambda\in\Lambda} a_\lambda \lambda \in D$, then $ord\ u = max\{ord\lambda \mid a_\lambda \neq 0\}$.

We may consider D as a filtered ring with the filtration $(D_\mu)_{\mu\in\mathbb{Z}}$ such that $D_\mu = \{u \in D \mid ord\ u \leq \mu\}$ for any $\mu \in \mathbb{N}$ and $D_\mu = 0$ for $\mu < 0$. It is clear that $\bigcup\{D_\mu \mid \mu \in \mathbb{Z}\} = D$, $D_\mu \subseteq D_{\mu+1}$ for any $\mu \in \mathbb{Z}$ and $D_\nu D_\mu = D_{\mu+\nu}$ for any $\mu, \nu \in \mathbb{Z}$.

DEFINITION 4.1. Let R be a Δ-Σ-field and M be a Δ-Σ-module. A sequence $(M_\mu)_{\mu\in\mathbb{Z}}$ of R-vector subspaces of the module M is called a filtration of M if the following three conditions hold:

(i) $M_\mu \subseteq M_{\mu+1}$ for all $\mu \in \mathbb{Z}$ and $M_\mu = 0$ for all sufficiently small $\mu \in \mathbb{Z}$.

(ii) $\bigcup\{M_\mu \mid \mu \in \mathbb{Z}\} = M$.

(iii)$D_\nu M_\mu \subseteq M_{\mu+\nu}$ for any $\mu \in \mathbb{Z}, \nu \in \mathbb{N}$.

If every R-vector space M_μ is finitely generated and there exist numbers $\mu_0 \in \mathbb{Z}$ such that $D_\nu M_\mu = M_{\mu+\nu}$ for all $\mu \geq \mu_0$, $\nu \in \mathbb{N}$, then the filtration $(M_\mu)_{\mu\in\mathbb{Z}}$ is called an excellent filtration of M.

EXAMPLE 4.1. Let M be a finitely generated Δ-Σ-module(i.e. a left D-module) with generators $h_1, \cdots, h_q$. If

$$M_\mu = D_\mu h_1 + \cdots + D_\mu h_q$$

for any $\mu \in \mathbb{Z}$, then $(M_\mu)_{\mu\in\mathbb{Z}}$ is an excellent filtration of M. □

DEFINITION 4.2. A polynomial $f(t_1, \cdots, t_p)$ in p variables $t_1, \cdots, t_p$ with rational coefficients is called numerical if $f(t_1, \cdots, t_p) \in \mathbb{Z}$ for all sufficiently large $(r_1, \cdots, r_p) \in \mathbb{Z}^p$, i.e. there exists a n-tuple $(s_1, \cdots, s_p) \in \mathbb{Z}^p$ such that $f(r_1, \cdots, r_p) \in \mathbb{Z}$ for all integers $r_1, \cdots, r_p \in \mathbb{Z}$ with $r_i \geq s_i$ $(1 \leq i \leq p)$.

The following theorem proved in Levin (2000) described the numerical polynomials associated with subsets of $\mathbb{N}^m \times \mathbb{Z}^n$.

THEOREM 4.1. Let A be a subset of $\mathbb{N}^m \times \mathbb{Z}^n$. Choose the canonical ortant decomposition of $\mathbb{Z}^n$ (see Example 2.1). Let $\trianglelefteq$ be the partial order on $\mathbb{N}^m \times \mathbb{Z}^n$ such that $(k_1, \cdots, k_m, l_1, \cdots, l_n) \trianglelefteq (r_1, \cdots, r_m, s_1, \cdots, s_n)$ if and only if $(l_1, \cdots, l_n)$ and $(s_1, \cdots, s_n)$ belong to a same ortant and

$$(r_1, \cdots, r_m, |s_1|, \cdots, |s_n|) \in \{(k_1, \cdots, k_m, |l_1|, \cdots, |l_n|) + \mathbb{N}^{m+n}\}.$$

Furthermore, let

$$W_A = \{w \in \mathbb{N}^m \times \mathbb{Z}^n \mid \textit{there is no element } a \in A \textit{ such that } a \trianglelefteq w\}$$

and

$$W_A[r, s] = \{(k_1, \cdots, k_m, l_1, \cdots, l_n) \in W_A \mid k_1 + \cdots + k_m \leq r, |l_1| + \cdots + |l_n| \leq s\}.$$

Then there exists a numerical polynomial $\psi_A(t_1, t_2)$ in two variables t_1 and t_2 with the following properties.

(i) $\psi_A(r, s) = Card\ W_A[r, s]$ for all sufficiently large $(r, s) \in \mathbb{N}^2$.

(ii) $deg\ \psi_A \leq m + n$, $deg_{t_1}\psi_A \leq m$, and $deg_{t_2}\psi_A \leq n$.

(iii)If $A = \emptyset$, then $deg\ \psi_A = m + n$. In this case,

$$\psi_A(t_1, t_2) = \binom{t_1 + m}{m} \sum_{i=0}^{n} (-1)^{n-i} 2^i \binom{n}{i} \binom{t_2 + i}{i}.$$

(iv)$\psi_A(t_1, t_2) = 0$ if and only if $(0, \cdots, 0) \in A$. □

The analog of Theorem 4.1 for the existence of numerical polynomial $\phi_A(t)$ in one variable t associated with the subset A of $\mathbb{N}^m \times \mathbb{Z}^n$ can be obtained in the same way as that used in the proof of Theorem 4.1.(see Levin 2000). We state it as follows.

COROLLARY. Let A, $\trianglelefteq$ and W_A be the same as in the conditions of Theorem 4.1. Let

$$W_A[\mu] = \{(k_1, \cdots, k_m, l_1, \cdots, l_n) \in W_A \mid k_1 + \cdots + k_m + |l_1| + \cdots + |l_n| \leq \mu\}.$$

Then there exists a numerical polynomial $\phi_A(t)$ with the following properties.
(i) $\phi_A(\mu) = Card\ W_A[\mu]$ for all sufficiently large $\mu \in \mathbb{N}$.
(ii) $deg\ \phi_A \leq m + n$, and if $A = \emptyset$ then $deg\ \phi_A = m + n$.
(iii)$\phi_A(t) = 0$ if and only if $(0, \cdots, 0) \in A$. □

DEFINITION 4.3. The numerical polynomial $\phi(t)$ is called difference-differential dimension polynomial in one variable t associated with M,if
(i) $deg(\phi(t)) \leq m + n$ and
(ii)$\phi(\mu) = dim_R M_\mu$ for all sufficiently large $\mu \in \mathbb{N}$.

Choose the canonical orthant decomposition on $\mathbb{Z}^n$ as in Example 2.1 and define the generalized term order "$\prec$" on ΛE of the terms of F as follows(see Example 2.2):

If $u = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} e_i$ and $v = \delta_1^{r_1} \cdots \delta_m^{r_m} \sigma_1^{s_1} \cdots \sigma_n^{s_n} e_j$, then

$$u \prec v \Longleftrightarrow (ord\ u, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n)$$

$$< (ord\ v, e_j, r_1, \cdots, r_m, |s_1|, \cdots, |s_n|, s_1, \cdots, s_n) \quad \textit{in lexicographic order.}$$

THEOREM 4.2. Let R be a Δ-Σ-field, D the ring of Δ-Σ-operators over R and M be a finitely generated Δ-Σ-module with generators $h_1, \cdots, h_q$. Let F be a free Δ-Σ-module with a basis $e_1, \cdots, e_q$ and $\pi :\ F \longrightarrow M$ the natural Δ-Σ-epimorphism of F onto M (i.e. $\pi(e_i) = h_i$ for $i = 1, \cdots, q$).

Let M_μ be the vector R-space as in Example 2.1. Suppose $G = \{g_1, \cdots, g_d\}$ is a Gröbner basis of $N = ker\ \pi$ with respect to the generalized term order "$\prec$" defined above, U_μ is the set of all terms $w \in \Lambda E$ such that $ord\ w \leq \mu$ and $w \neq lt(\lambda g_i)$, $\lambda \in \Lambda$, $i = 1, \cdots, d$. Then $\pi(U_\mu)$ is a basis of the R-vector space M_μ.

PROOF. First, we need to show that the set $\pi(U_\mu)$ generates the R-vector space M_μ. Since $M_\mu = D_\mu h_1 + \cdots + D_\mu h_q$, it is enough to show that every element λh_i ($i = 1, \cdots, q$, $\lambda \in \Lambda$, $ord\ \lambda \leq \mu$), that does not belong to $\pi(U_\mu)$, can be written as a finite linear combination of elements of $\pi(U_\mu)$ with coefficients from R. Since $\lambda h_i \notin \pi(U_\mu)$ implies $\lambda e_i \notin U_\mu$, whence $\lambda e_i = lt(\lambda'\ g_j)$ for some $\lambda' \in \Lambda$, $g_j \in G$. Therefore

$$\lambda'\ g_j = a_j \lambda e_i + \sum_\nu a_\nu \lambda_\nu e_\nu$$

where $a_j \neq 0$ and $a_\nu \neq 0$ for finitely many a_ν. Obviously, $\lambda_\nu e_\nu \prec \lambda e_i$ and then $ord\ \lambda_\nu \leq \mu$. Note that $G \subseteq N = ker(\pi)$, we have $0 = \pi(g_j)$ and

$$0 = \lambda' \pi(g_j) = \pi(\lambda'\ g_j) = a_j \pi(\lambda e_i) + \sum_\nu a_\nu \pi(\lambda_\nu e_\nu) = a_j \lambda h_i + \sum_\nu a_\nu \lambda_\nu h_\nu.$$

So that λh_i is a finite linear combination with coefficients from R of some elements of the form $\lambda_\nu h_\nu$ $(1 \leq \nu \leq q)$ such that $ord\ \lambda_\nu \leq \mu$ and $\lambda_\nu e_\nu \prec \lambda e_i$.

Thus, we can apply the induction on λe_j ($\lambda \in \Lambda$, $1 \leq j \leq q$) with respect to the order "$\prec$" and obtain that every element λh_i ($ord\ \lambda \leq \mu$, $1 \leq i \leq q$) can be written as a finite linear combination of elements of $\pi(U_\mu)$ with coefficients from R.

Now, let us prove that the set $\pi(U_\mu)$ is linearly independent over R. Suppose that $\sum_{i=1}^{l} a_i \pi(u_i) = 0$ for some $u_1, \cdots, u_l \in U_\mu$, $a_1, \cdots, a_l \in R$. Then $h = \sum_{i=1}^{l} a_i u_i \in N$ and $lt(h) \neq lt(\lambda g_i)$, $\lambda \in \Lambda$, $i = 1, \cdots, d$, by the definition of U_μ. Since G is a Gröbner basis of N it is follows from Proposition 3.2 (iii) that $h = 0$. Therefore $a_1 = \cdots = a_l = 0$. This completes the proof of the theorem. □

From Theorem 4.2 the difference-differential dimension polynomial $\phi(t)$ (with $\phi(\mu)$ is the dimension of M_μ as an R-vector space) can be computed by difference-differential Gröbner bases.

THEOREM 4.3. Let R be a Δ-Σ-field, D the ring of Δ-Σ-operators over R and M be a finitely generated Δ-Σ-module, and $(M_\mu)_{\mu \in \mathbb{Z}}$ an excellent filtration of M. If $G = \{g_1, \cdots, g_d\}$ is a Gröbner basis of $N = ker\ \pi$ as theorem 4.2, then the difference-differential dimension polynomial $\phi(t)$ such that $deg(\phi(t)) \leq m + n$ and $\phi(\mu) = dim_R M_\mu$ for all sufficiently large $\mu \in \mathbb{N}$ can be computed as

$$\phi(\mu) = Card(U_\mu) = Card\{w \in \Lambda E \mid ord\ w \leq \mu;\ w \neq lt(\lambda g_i), \lambda \in \Lambda, g_i \in G\}$$

PROOF. Since $(M_\mu)_{\mu \in \mathbb{Z}}$ is an excellent filtration of M it follows that every M_μ is a finitely generated R-vector space and $D_\nu M_\mu = M_{\mu+\nu}$ for $\mu \geq \mu_0$, $\nu \geq 0$. Let $h_1, \cdots, h_q$ be a basis of the R-vector space M_{μ_0}. Then the elements $h_1, \cdots, h_q$ generate M as a left D-module and $M_\mu = \sum_{i=1}^{q} D_{\mu-\mu_0} h_i$ for all $\mu \geq \mu_0$. Without loss of generality we can assume that $\mu_0 = 0$.(If $\phi(t)$ is a numerical polynomial with the desired properties that corresponds to the case $\mu_0 = 0$ then $\phi(t - \mu_0)$ is the one for arbitrary $\mu_0 \in \mathbb{Z}$.) Thus we may suppose that $M = \sum_{i=1}^{q} D h_i$ and $M_\mu = \sum_{i=1}^{q} D_\mu h_i$ for all $\mu \in \mathbb{Z}$.

Let F be a free Δ-σ-module with a basis $e_1, \cdots, e_q$. Let $\pi : F \longrightarrow M$, $N = ker\ \pi$ and $U_\mu (\mu \in \mathbb{N})$ be the same as in the conditions of Theorem 4.2. Furthermore, let "$\prec$" be the generalized term order on ΛE of the terms of F and $G = \{g_1, \cdots, g_d\}$ be the Gröbner basis of N as in Theorem 4.2. By Theorem 4.2, for any $\mu \in \mathbb{N}$, $\pi(U_\mu)$ is a basis of the R-vector space M_μ. Note that in the second part of the proof of Theorem 4.2 it was shown that the restriction of π on U_μ is bijective, we have $dim_R M_\mu = Card\ \pi(U_\mu) = Card(U_\mu)$.

Note that $U_\mu = \{w \in \Lambda E \mid ord\ w \leq \mu;\ w \neq lt(\lambda g_i), \lambda \in \Lambda, g_i \in G\}$. Let $V_i^{(j)}$ be a finite set of generators of the $R[\Lambda_j]$-module ${}_{R[\Lambda_j]}\langle lt(\lambda g_i) \in \Lambda_j E \mid \lambda \in \Lambda \rangle$. Let $V = \bigcup_{i,j} V_i^{(j)}$. Then $U_\mu = \{w \in \Lambda E \mid ord\ w \leq \mu;\ w$ is not a multiple of any element $v \in V\}$.

Let $V_{e_i} = \{v \in V \mid v = \lambda e_i, \lambda \in \Lambda\}$ and $U_\mu^{(i)} = \{w \in \Lambda e_i \mid ord\ w \leq \mu;\ w$ is not a multiple of any element $v \in V_{e_i}\}$, $i = 1, \cdots, q$. Then $Card(U_\mu) = \sum_{i=1}^{q} Card(U_\mu^{(i)})$. By The corollary of theorem 4.1, there exists a numerical polynomial $\phi_i(t)$ such that $deg(\phi_i(t)) \leq m + n$ and $\phi_i(\mu) = Card(U_\mu^{(i)})$, $i = 1, \cdots, q$, for all sufficiently large $\mu \in \mathbb{N}$. Therefore $\phi(t) = \sum_{i=1}^{q} \phi_i(t)$ satisfies that $deg(\phi(t)) \leq m + n$ and $\phi(\mu) = Card(U_\mu) = dim_R M_\mu$ for all sufficiently large $\mu \in \mathbb{N}$. □

EXAMPLE 4.2. Let R be a difference-differential field whose basic sets Δ and Σ consist of a single derivation operator δ and a single automorphism σ, respectively. Furthermore, let D be the ring of Δ-Σ-operators over R and $M = Dh$ be a cyclic Δ-Σ-module whose generator h satisfies the defining equation

$$(\delta^a\sigma^b + \delta^a\sigma^{-b} + \delta^{a+b})h = 0$$

where $a, b \in \mathbb{N}$. In other words, M is isomorphic to the factor module of a free Δ-Σ-module F with a free generator e by its Δ-Σ-submodule $N = D(\delta^a\sigma^b + \delta^a\sigma^{-b} + \delta^{a+b})e$. Let the generalized term order $\prec$ on ΛE is the same as in Theorem 4.3. Then $\{g = (\delta^a\sigma^b + \delta^a\sigma^{-b} + \delta^{a+b})e\}$ is a difference-differential Gröbner basis of N(If W is generated by one element $g \in F\backslash\{0\}$, then any finite subset G of $W\backslash\{0\}$ containing g is a difference-differential Gröbner basis of W). since $lt(g) = (\delta^{a+b})e$ belong to any orthant of ΛE, it follows that $lt(\lambda g) = \lambda(\delta^{a+b})e$ for any $\lambda \in \Lambda$. Then by Theorem 4.3,

$$dim_R M_t = Card(U_t) = Card\{u \in \Lambda \mid ord\ u \leq t; u \neq \lambda\delta^{a+b}, \lambda \in \Lambda\}.$$

Therefore,

$$dim_R M_t = Card\{\delta^c\sigma^d \mid c \in \mathbb{N}, d \in \mathbb{Z}, c + |d| \leq t, (c, |d|) \notin \{(a + b, 0) + \mathbb{N}^2\}\}$$

$$= Card\{\delta^c\sigma^d \mid c \in \mathbb{N}, d \in \mathbb{Z}, c + |d| \leq t\}$$

$$-Card\{\delta^c\sigma^d \mid c \in \mathbb{N}, d \in \mathbb{Z}, c + |d| \leq t - (a + b)\}$$

$$= [(t + 2)(t + 1) - (t + 1)] - [(t - a - b + 2)(t - a - b + 1) - (t - a - b + 1)]$$

$$= 2(a + b)t + (a + b)(2 - a - b).$$

□

The next example shows that even we choose another generalized term order the the difference-differential dimension polynomial can be also computed.

EXAMPLE 4.3. Let M be the Δ-Σ-module same as in Example 4.2. But the generalized term order $\prec$ on ΛE is defined as follows:

$$\delta^k\sigma^l e \prec \delta^r\sigma^s e \Longleftrightarrow (k + |l|, |l|, k, l) < (r + |s|, |s|, r, s) \ \ \textit{in lexicographic order.}$$

Note that Theorem 4.2 and 4.3 still valid for "$\prec$". Denote $\{\delta^k\sigma^l \mid l \geq 0\}$ by Λ_1 and $\{\delta^k\sigma^l \mid l \leq 0\}$ by Λ_2. Since $lt(g) = \delta^a\sigma^b e \in \Lambda_1$ and $lt(\sigma^{-1}g) = \delta^a\sigma^{-(b+1)}e \in \Lambda_2$ it follows that

$$\{lt(\lambda g) \in \Lambda_1 \mid \lambda \in \Lambda\} = \Lambda_1\delta^a\sigma^b e \qquad \{lt(\eta g) \in \Lambda_2 \mid \eta \in \Lambda\} = \Lambda_2\delta^a\sigma^{-(b+1)}e.$$

Therefore

$$dim_R M_t = Card\{\delta^c\sigma^d \mid c, d \in \mathbb{N}, c + d \leq t, (c, d) \notin \{(a, b) + \mathbb{N}^2\}\} +$$

$$+Card\{\delta^c\sigma^d \mid c \in \mathbb{N}, d \in \mathbb{Z}, d < 0, c + |d| \leq t, (c, -d) \notin \{(a, b + 1) + \mathbb{N}^2\}\}$$

$$= [\frac{1}{2}(t + 1)(t + 2) - \frac{1}{2}(t - a - b + 1)(t - a - b + 2)] +$$

$$+[\frac{1}{2}t(t+1) - \frac{1}{2}(t-a-b)(t-a-b+1)]$$
$$= 2(a+b)t + (a+b)(2-a-b).$$

The result coincide with that of example 4.2. □

EXAMPLE 4.4. Let R be a difference-differential field whose basic sets Δ and Σ consist of two automorphism $\sigma_1,\ \sigma_2$. Furthermore, let D be the ring of Δ-Σ-operators over R and $M = Dh$ be a cyclic Δ-Σ-module whose generators h satisfies the defining equation

$$(\sigma_1\sigma_2(\sigma_1+1))h = 0$$

$$(\sigma_1\sigma_2(\sigma_2+1))h = 0$$

In other words, M is isomorphic to the factor module of a free Δ-Σ-module F with a free generator e by its Δ-Σ-submodule N which have generators $\{f = (\sigma_1\sigma_2(\sigma_1+1)),\ g = (\sigma_1\sigma_2(\sigma_2+1))\}$. Let the generalized term order $\prec$ on ΛE is the same as in Theorem 4.3. Then example 3.2 shows that

$$\{g_1, g_2, g_3, g_4\} = \{\sigma_2 + 1, \sigma_1 + 1, {\sigma_1}^{-1} + 1, {\sigma_2}^{-1} + 1\}$$

is a Gröbner basis of N.

Therefore

$$dim_R M_t = 1$$

which means that

$$dim_R M = 1$$

and then the solution space of the system of difference equations

$$(\sigma_1\sigma_2(\sigma_1+1))z = 0$$

$$(\sigma_1\sigma_2(\sigma_2+1))z = 0$$

have dimension 1 over R. □

The above examples show that the computation of difference-differential dimension polynomial by difference-differential Gröbner basis is more direct and simpler than traditional Gröbner basis method.

5 Difference-differential dimension polynomials in two variables

Now we consider difference-differential dimension polynomials $\psi_A(t_1, t_2)$ in two variables t_1 and t_2 by the approach of difference-differential Gröbner bases.

Choose the canonical orthant decomposition on $\mathbb{Z}^n$ as in Example 2.1 and define the generalized term order "$\prec$" and "$\prec'$" on Λe of the terms of F as follows:

If $u = \delta_1^{k_1}\cdots\delta_m^{k_m}\sigma_1^{l_1}\cdots\sigma_n^{l_n}e_i$ and $v = \delta_1^{r_1}\cdots\delta_m^{r_m}\sigma_1^{s_1}\cdots\sigma_n^{s_n}e_j$, let

$$|u|_1 = k_1 + \cdots + k_m$$

$$|u|_2 = |l_1| + \cdots + |l_n|$$

and

$$u \prec v \Longleftrightarrow (|u|_2, |u|_1, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n)$$

$$< |v|_2, |v|_1, e_j, r_1, \cdots, r_m, |s_1|, \cdots, |s_n|, s_1, \cdots, s_n) \ \textit{in lexicographic order.}$$

$$u \prec' v \Longleftrightarrow (|u|_1, |u|_2, e_i, k_1, \cdots, k_m, |l_1|, \cdots, |l_n|, l_1, \cdots, l_n)$$

$$< (|v|_1, |v|_2, e_j, r_1, \cdots, r_m, |s_1|, \cdots, |s_n|, s_1, \cdots, s_n) \ \textit{in lexicographic order.}$$

If $u = \sum_{\lambda \in \Lambda} a_\lambda \lambda \in D$, then $|u|_1 = max\{|\lambda|_1 \mid a_\lambda \neq 0\}$, $|u|_2 = max\{|\lambda|_2 \mid a_\lambda \neq 0\}$.

We may consider D as a bifiltered ring with the bifiltration $(D_{rs})_{r,s\in\mathbb{Z}}$ such that $D_{rs} = \{u \in D \mid |u|_1 \leq r, \ |u|_2 \leq s\}$ for any $r, s \in \mathbb{N}$ and $D_{rs} = 0$ if at least one of the numbers r, s is negative. Obviously $\bigcup\{D_{rs} \mid r, s \in \mathbb{Z}\} = D$, $D_{rs} \subseteq D_{r+1,s}$, $D_{rs} \subseteq D_{r,s+1}$ for any $r, s \in \mathbb{Z}$ and $D_{kl}D_{rs} = D_{r+k,s+l}$ for any $r, s, k, l \in \mathbb{Z}$.

Let M be a finitely generated left D-module with generators $h_1, \cdots, h_q$. let

$$M_{rs} = D_{rs}h_1 + \cdots + D_{rs}h_q$$

for any $r, s \in \mathbb{Z}$, then $(M_{rs})_{r,s\in\mathbb{Z}}$ is an excellent bifiltration of M.

DEFINITION 5.1. The numerical polynomial $\psi(t_1, t_2)$ is called difference-differential dimension polynomial in two variables t_1 and t_2 associated with M,if
(i) $deg\ \psi \leq m + n$, $deg_{t_1}\psi \leq m$, and $deg_{t_2}\psi \leq n$ and
(ii)$\psi(t_1, t_2) = dim_R M_{t_1,t_2}$ for all sufficiently large $t_1, t_2 \in \mathbb{N}$.

Levin(2000) investigated the difference-differential dimension polynomials in two variables by characteristic set method. The method of Levin is rather delicate but no algorithm for compute the characteristic set. So in Levin(2000) there are examples of computing the difference-differential dimension polynomials in two variables just for N is cyclic module. We will show that, by the method of difference-differential Gröbner bases the same result can be obtained.

THEOREM 5.1. Let R be a Δ-Σ-field, D and M be as above. M have generators $h_1, \cdots, h_q$. Let F be a free Δ-Σ-module with a basis $e_1, \cdots, e_q$ and $\pi : \ F \longrightarrow M$ the natural Δ-Σ-epimorphism of F onto M ($\pi(e_i) = h_i$ for $i = 1, \cdots, q$).

Let $\prec$ and $\prec'$ be the generalized term orders on Λe of the terms of F defined above. Suppose that $N = ker\ \pi$ is a cyclic module and $G = \{g\}$ is a reduced Gröbner basis of N with respect to $\prec$. Let

$$U_{r,s} = \{w \in \Lambda e | |w|_1 \leq r, |w|_2 \leq s;$$

$$w \neq lt_\prec(\lambda g_i), g_i \in G \quad or \quad w = lt_\prec(\lambda g_j) : |[lt_{\prec'}(\lambda g_j)]|_1 > r\}$$

Then the difference-differential dimension polynomial $\psi(t_1, t_2)$ in two variables t_1 and t_2 associated with M can be computed by the following:

$$\psi(r, s) = Card\ U_{r,s} = Card\ \{w \in \Lambda e | |w|_1 \leq r, |w|_2 \leq s; w \neq lt_\prec(\lambda g_i), g_i \in G\}$$

$$+Card\ \{w \in \Lambda e||w|_1 \leq r, |w|_2 \leq s; w = lt_{\prec}(\lambda g_j) : |[lt_{\prec'}(\lambda g_j)]|_1 > r\}, g_i \in G\}$$

PROOF. First, similar to the proof of Theorem 4.2, let us show that every element λh_i ($i = 1, \cdots, q$, $\lambda \in \Lambda$, $|\lambda|_1 \leq r$, $|\lambda|_2 \leq s$), that does not belong to $\pi(U_{r,s})$, can be written as a finite linear combination of elements of $\pi(U_{r,s})$ with coefficients from R. Since $\lambda h_i \notin \pi(U_{r,s})$ implies $\lambda e_i \notin U_{r,s}$, whence $\lambda e_i = lt_{\prec}(\lambda' g_j)$ for some $\lambda' \in \Lambda$, $g_j \in G$, and $|[lt_{\prec'}(\lambda' g_j)]|_1 \leq r$. Therefore

$$\lambda' g_j = a_j \lambda e_i + \sum_{\nu} a_\nu \lambda_\nu e_\nu$$

where $a_j \neq 0$ and $a_\nu \neq 0$ for finitely many a_ν. Obviously, $\lambda_\nu e_\nu \prec \lambda e_i = lt_{\prec}(\lambda' g_j)$. Then by the definition of $\prec$, $|\lambda_\nu|_2 \leq s$. On the other hand, since $|[lt_{\prec'}(\lambda' g_j)]|_1 \leq r$ and $\lambda_\nu e_\nu \prec' lt_{\prec'}(\lambda' g_j)$, it follows from the definition of $\prec'$ that $|\lambda_\nu|_1 \leq r$. Now note that $G \subseteq N = ker(\pi)$, we have $0 = \pi(g_j)$ and

$$0 = \lambda' \pi(g_j) = \pi(\lambda' g_j) = a_j \pi(\lambda e_i) + \sum_{\nu} a_\nu \pi(\lambda_\nu e_\nu) = a_j \lambda h_i + \sum_{\nu} a_\nu \lambda_\nu h_\nu.$$

So that λh_i is a finite linear combination with coefficients from R of some elements of the form $\lambda_\nu h_\nu$ ($1 \leq \nu \leq q$) such that $|\lambda_\nu|_1 \leq r$, $|\lambda_\nu|_2 \leq s$ and $\lambda_\nu e_\nu \prec \lambda e_i$.

If there are some $\lambda_\nu h_\nu \notin \pi(U_{r,s})$, then we may repeat the same procedure on $\lambda_\nu h_\nu$ as that on λh_i. Thus, we can apply the induction on λe_j ($\lambda \in \Lambda$, $1 \leq \nu \leq q$) with respect to the order $\prec$ and obtain that

$$\lambda h_i = \sum_{\mu} b_\mu \lambda_\mu h_\mu$$

such that $|\lambda_\mu|_1 \leq r$, $|\lambda_\mu|_2 \leq s$ and $\lambda_\mu h_\mu \in \pi(U_{r,s})$.

Now we must prove that the set $\pi(U_{r,s})$ is linearly independent over R. Suppose that $\sum_{i=1}^{l} a_i \pi(u_i) = 0$ for some $u_1, \cdots, u_l \in U_{r,s}$, $a_1, \cdots, a_l \in R$. Then $h = \sum_{i=1}^{l} a_i u_i \in N$. Note that $N = Dg$ and $G = \{g\}$ is the reduced Gröbner basis of N. If $lt_{\prec}(h) \neq lt(\lambda g)$, $\lambda \in \Lambda$, then it is follows from Gröbner basis properties that $h = 0$. Therefore $a_1 = \cdots = a_l = 0$. If $lt_{\prec}(h) = lt_{\prec}(\mu g)$ and $|[lt_{\prec'}(\mu g)]|_1 > r$, then by Zhou and Winkler(2006) Proposition 3.1, there is a unique term w_1 of g such that $lt_{\prec}(h) = lt_{\prec}(\mu g) = \mu w_1$. But we have also $h = pg$, $p \in D$, then there is a unique term λ_1 of p and w_2 of g such that $lt_{\prec}(h) = lt_{\prec}(pg) = \lambda_1 w_2$. Then by Zhou and Winkler(2006) Lemma 3.3, we have $w_1 = w_2$ and then $\lambda_1 = \mu$. Therefore all terms of μg are in pg. Because the $|u|_1$ means the grad of differential operators, it is easy to check that $|pg|_1 = |\mu g|_1 > r$. This is a contradiction with that $|h|_1 \leq r$. This means $h = 0$ and then $\pi(U_{r,s})$ is linearly independent over R. Actually π is bijection on $U_{r,s}$. So

$$\psi(r,s) = dim_R M_{r,s} = Card\ U_{r,s}$$

This completes the proof of the theorem. □

EXAMPLE 5.1. Let R be a difference-differential field whose basic sets Δ and Σ consist of a single δ and a single σ. Furthermore, let D be the ring of Δ-Σ-operators over R and $M = Dh$ be a cyclic Δ-Σ-module whose generators h

satisfies the defining equation

$$(\delta\sigma + \sigma^{-2})h = 0$$

In other words, M is isomorphic to the factor module of a free Δ-Σ-module F with a free generator e by its Δ-Σ-submodule N which have generators $\{g = \delta\sigma + \sigma^{-2}\}$. We compute the difference-differential dimension polynomial $\psi(r, s)$.

Now N is a cyclic submodule of F, by theorem 5.1, we have to compute the GB of N w.r.t $\prec$ and $Card\ U_{r,s}$.

Clearly the GB is $\{g = \delta\sigma + \sigma^{-2}\}$ and $lt(g) = \sigma^{-2} \in \Lambda_2$. Because $\sigma g = \delta\sigma^2 + \sigma^{-1}$ which leading term is $\delta\sigma^2 \in \Lambda_1$, we have

$$lt(\lambda g) = \Lambda_1\delta\sigma^2 \bigcup \Lambda_2\sigma^{-2}$$

Put

$$U'_{r,s} = \{w \in \Lambda e || w|_1 \leq r, |w|_2 \leq s; w \neq lt_{\prec}(\lambda g)\}$$

$$U''_{r,s} = \{w \in \Lambda e || w|_1 \leq r, |w|_2 \leq s; w = lt_{\prec}(\lambda g) : |[lt_{\prec'}(\lambda g)]|_1 > r\}$$

Then

$$Card\ U_{r,s} = Card\ U'_{r,s} + Card\ U''_{r,s}$$

and

$$\psi(t_1, t_2) = dim_R M_{t_1,t_2} = Card\ U_{t_1,t_2} = (3t_1 + t_2 + 2) + (t_2 - 1) = 3t_1 + 2t_2 + 1$$

□

References

Buchberger, B. (1985). Gröbner bases: An algorithmic method in polynomial ideal theory. In *Multidimensional Systems Theory*, pp. 184-232. D. Reidel.

Carra Ferro, G. (1989). *Gröbner bases and Differential Algebra.* LNCS, **356**, pp.129-140.

Carra Ferro, G. (1997). Differential Gröbner bases in one variable and in the partial case. Algorithms and software for symbolic analysis of nonlinear systems. *Math. Comput. Modelling*, **25**, 1-10.

Insa, M., Pauer, F. (1998). Gröbner bases in rings of differential operators. In *Gröbner Bases and Applications*, pp. 367-380. New York, Cambridge University Press.

Johnson, J. (1974). Kahler differentials and differential algebra in arbitrary characteristic. *Trans. Am. Math. Soc.*, **192**, 201-208.

Kolchin, E. R. (1964). The notion of dimension in the theory of algebraic differential equations, *Bull. Am. Math. soc.*,**70**, 570-573.

Kondrateva, M. V., Levin, A. B., Mikhalev, A. V., Pankratev, E. V. (1992). Computation of dimension polynomials. *Int. J. Algebra Comput.*,**2**, 117-137.

Kondrateva, M. V., Levin, A. B., Mikhalev, A. V., Pankratev, E. V. (1999). *Differential and Difference Dimension Polynomials.* Dordrecht, Kluwer Academic Publishers.

Levin, A. B., Mikhalev, A. V.(1987). Differential dimension polynomial and the strength of a system of differential equations. In *Computable Invariants in the Theory of Algebraic Systems. Collection of Papers*, pp. 58-66. Novosibirsk. In Russian.

Levin, A. B.(1978). Characteristic polynomials of filtered difference modules and of difference field extensions.*Russ. Math Surv.*,**33**,165-166.

Levin, A. B.(2000). Reduced Gröbner bases, free difference-differential modules and difference-differential dimension polynomials.*J. Symb. Comput.*, **30**, 357-382.

Mikhalev, A. V., Pankratev, E. V. (1989). *Computer Algebra. Computations in Differential and Difference Algebra.* Moscow, Moscow State Univ. Press, In Russian.

Noumi, M. (1988). Wronskima determinants and the Gröbner representation of linear differential equation. In *Algebraic Analysis*, pp. 549-569. Boston, Academic Press.

Nordbeck, P.(1998). On some basic applications of Gröbner bases in noncommutative polynomial rings. In *Gröbner Bases and Applications*, pp.463-472. New York, Cambridge University Press.

Oaku, T., Shimoyama, T.(1994). A Gröbner basis method for modules over rings of differential operators. *J. Symb. Comput.*, **18**, 223-248.

Pauer, F., Zampieri, S. (1996). Gröbner bases with respect to generalized term orders and their applications to the modelling problem. *J. Symb. Comput.*, **21**, 155-168.

Pauer, F., Unterkircher, A.(1999). Gröbner bases for ideals in Laurent polynomial rings and their applications to systems of difference equations. *AAECC*, **9**,271-291.

Takayama, N. (1989). Gröbner basis and the problem of contiguous relations. *Japan J. Appl. Math.*, **6**, 147-160.

Zhou, M., Winkler, F. (2006). Gröbner bases in difference-differebtial modules. In *Proceedings of the 2006 ISSAC*, pp ACM Press.