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A Lie algebra method for rational parametrization of Severi–Brauer surfaces

Willem A. de Graaf^{a,*}, Michael Harrison^b, Jana Pílníková^a,
Josef Schicho^a

^a RICAM, Linz, Austria

^b University of Sydney, Australia

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Abstract

It is well known that a Severi–Brauer surface has a rational point if and only if it is isomorphic to the projective plane. Given a Severi–Brauer surface, we study the problem to decide whether such an isomorphism to the projective plane, or such a rational point, does exist; and to construct such an isomorphism or such a point in the affirmative case. We give an algorithm using Lie algebra techniques. The algorithm has been implemented in Magma.

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1. Introduction

The problem considered in this paper is to decide whether a given surface is isomorphic to the projective plane over the rational numbers and if so, to find an isomorphism. It is easy to decide this over the complex numbers (see Section 2). Hence we can assume that the surface is a twist of \mathbb{P}^2 , also called a Severi–Brauer surface.

The problem comes from the parametrization of the surfaces. When trying to parametrize a surface over the rational numbers, one can reduce to several base cases [23]. The

* Corresponding author.

E-mail address: willem.degraaf@oeaw.ac.at (W.A. de Graaf).

Severi–Brauer surface arises as one of them. Therefore our problem appears as a subproblem of finding a parametrization of a surface with rational coefficients.

Parameterizing a Severi–Brauer surface (i.e., finding an isomorphism to \mathbb{P}^2) is equivalent to finding a rational point on the surface: from an isomorphism one can construct all points, and in the other direction one can construct an isomorphism when a single point is known [19]. This fact is not used in our paper.

There is a well-known correspondence between Severi–Brauer surfaces and central simple algebras of degree 3 (cf. [17]). The split Severi–Brauer surfaces (i.e., those isomorphic to the projective plane) correspond to the split central simple algebras (those isomorphic to the full matrix algebra). There are many classical number-theoretical results available which are useful for deciding whether the given central simple algebra is split or not. Here we reduce the problem to the case of cyclic algebra—this is possible because of a result of Wedderburn (cf. [17])—and solve a norm equation [6,20].

There are known constructions of the Severi–Brauer surface corresponding to a given central simple algebra. However, in the other direction there are no constructions available. We introduce an intermediate step: the Lie algebra of the given surface. This is the Lie algebra of the group of automorphisms of the surface.

Incidentally this is also the Lie algebra of regular vector fields. The relation between vector fields and the Lie algebra of the group of automorphisms has been mentioned in [10] for the affine and local analytic case.

The whole algorithm has been implemented in Magma [2]. The most expensive step is the solution of the norm equation.

The paper is formulated for rational numbers but anything written generalizes to number fields. In the last step one needs to solve a norm equation over the number field which is also implemented in Magma.

Most parts of the method can be extended to Severi–Brauer varieties of any dimension. In particular we can construct the corresponding central simple algebra. However, the construction of the cyclic algebra and the norm equation does not generalize.

The paper is structured as follows. In Section 2 we reduce the given surface to a Del Pezzo surface of degree 9. The isomorphism, if it exists, is then a linear projective map to the anticanonical embedding of \mathbb{P}^2 . In Section 3 we reduce the problem to finding an isomorphism of a given Lie algebra and the Lie algebra $\mathfrak{sl}_3(\mathbb{Q})$. In Section 4 we reduce the problem to finding an isomorphism of a given central simple algebra and the full matrix algebra $M_3(\mathbb{Q})$. In Section 5 we describe how to solve this problem by reducing to a norm equation.

The paper is carefully written such that the sections can be read in an arbitrary order. They are independent of each other.

2. The anticanonical embedding

We start with a projective surface (i.e., a variety of dimension 2) S over the rational numbers. Our goal is to decide whether S is isomorphic to \mathbb{P}^2 ; and if yes, we want to construct an isomorphism.

A first obvious necessary condition for S to be isomorphic to \mathbb{P}^2 is that S be nonsingular. This can be checked easily by the Jacobian criterion (see [9, Theorem I.5.3]). So we assume from now on that S is nonsingular.

For any nonsingular variety X , the *anticanonical bundle* \mathcal{A} is the determinant bundle of the tangent bundle TX (see [19]). In the surface case, this is just the antisymmetric tensor bundle of two times the tangent bundle. It is clear that \mathcal{A} is always a line bundle.

For any line bundle E over a projective variety X , the vector space $\Gamma(X, E)$ of sections is finitely generated over the ground field (which is \mathbb{Q} in our case). If $\dim(\Gamma(X, E)) =: n > 0$, then there exists an *associated rational map* $a_E : X \rightarrow \mathbb{P}^{n-1}$, which maps the point p to the ratio $(s_1(p) : \dots : s_n(p))$, where $\{s_1, \dots, s_n\}$ is a basis of $\Gamma(X, E)$. It is defined on the complement of the subset of X where all global sections vanish.

The definition of the associated map depends on the choice of the basis, but in a transparent way: if we choose a different basis, we get a projectively equivalent map.

The anticanonical map is the rational map associated to \mathcal{A} . For the explicit construction of the anticanonical map, we refer to [4].

Proposition 2.1. *The vector space of sections of the anticanonical bundle of \mathbb{P}^2 has dimension 10. The anticanonical map is an embedding. It is given by*

$$(s : t : u) \mapsto (x_0 : \dots : x_9) = (s^3 : t^3 : u^3 : s^2t : t^2u : u^2s : st^2 : tu^2 : us^2 : stu). \quad (1)$$

The image is a surface S_0 of degree 9, whose ideal is generated by 27 quadric polynomials.

Proof. The anticanonical bundle is isomorphic to $\mathcal{O}(3)$ (see [9, Example II.8.20]). The sections correspond to cubic forms, of which the above is a basis.

The ideal of the image is generated by the kernel of the linear evaluation map from the quadric forms in x_0, \dots, x_9 to the forms of degree 6 in s, t, u . This kernel has dimension $55 - 28 = 27$. \square

Recall that a projective surface $S \subset \mathbb{P}^n$ is called a *Del Pezzo surface* iff it is anticanonically embedded (see [19]). It is well known that $3 \leq n \leq 9$ in this case, and that the degree of the surface is then also n .

Theorem 2.2. *If S is isomorphic to \mathbb{P}^2 , then the anticanonical map $a_{\mathcal{A}}$ is an embedding, and the image is projectively equivalent to S_0 .*

Proof. An isomorphism $f : \mathbb{P}^2 \rightarrow S$ induces an isomorphism of the anticanonical bundles and a vector space isomorphism of global sections. \square

If X, Y are varieties defined over \mathbb{Q} , then we say that X is a *twist* of Y iff $X \otimes \mathbb{C}$ is \mathbb{C} -isomorphic to $Y \otimes \mathbb{C}$ (see [21]). Moreover, if $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^n$ are projective varieties, $n > 0$, then we say that X is a *projective twist* of Y iff X and Y are projectively equivalent.

If S is a twist of \mathbb{P}^2 , then its anticanonical embedding $a_{\mathcal{A}}(S)$ is a projective twist of S_0 , by the complex version of Theorem 2.2. The following theorem makes it possible to decide whether this statement holds or not.

Theorem 2.3. *S is a twist of \mathbb{P}^2 iff $\dim(\Gamma(S, \mathcal{A})) = 10$ and the anticanonical map is an embedding.*

Proof. It suffices to prove in the complex case that S is isomorphic to \mathbb{P}^2 if and only if $\dim(\Gamma(S, \mathcal{A})) = 10$ and the anticanonical map is an embedding. The “only if” direction follows from Proposition 2.1. Conversely, the statement that the anticanonical map is an embedding implies that S is a Del Pezzo surface (see [7]). By the classification of Del Pezzo surfaces (see [19]), the equality $\dim(\Gamma(S, \mathcal{A})) = 10$ implies that S is a Severi–Brauer surface. \square

Assume that $\dim(\Gamma(S, \mathcal{A})) = 10$. Then we know that S is a twist of \mathbb{P}^2 , and its anticanonical embedding $S_1 := a_{\mathcal{A}}(S)$ is a projective twist of S_0 . Moreover, S is equivalent to \mathbb{P}^2 iff S_1 is projectively equivalent to S_0 . Clearly, every projective transformation $S_0 \rightarrow S_1$ can be composed with the inverse of the anticanonical embedding to give an isomorphism $\mathbb{P}^2 \rightarrow S$. Hence, we have reduced the original problem to deciding whether a given Del Pezzo surface of degree 9 is projectively equivalent to S_0 , and to compute a projective transformation in the affirmative case.

3. The Lie algebra of a Severi–Brauer surface

In this section we start preparing the ground for the algorithm that establishes whether a given Severi–Brauer surface has a rational parametrization. We let S_0 be the Severi–Brauer surface given by the standard embedding of \mathbb{P}^2 (1). Furthermore, S will be an arbitrary Severi–Brauer surface anticanonically embedded into \mathbb{P}^9 . By Theorem 2.2, S and S_0 are isomorphic over \mathbb{Q} exactly if there is a matrix M such that

$$M \in \mathrm{GL}_{10}(\mathbb{Q}) \quad \text{and} \quad p \mapsto Mp \text{ is a bijection from } S_0 \text{ to } S. \quad (2)$$

Finding an isomorphism of a given surface S with S_0 and hence a parametrization of S therefore means finding M such that (2) holds.

For the moment we work over an arbitrary field $F \subset \mathbb{C}$. Let the anticanonically embedded Severi–Brauer surface S have a point over F and hence be isomorphic to $\mathbb{P}^2(F)$. Then its automorphism group $\mathrm{Aut}(S)$ is isomorphic to the automorphism group of $\mathbb{P}^2(F)$, which is $\mathrm{PGL}_3(F)$ (cf. [8]). In particular S admits only linear automorphisms, so $\mathrm{Aut}(S)$ consists of all $g \in \mathrm{PGL}_{10}(F)$ such that $gp \in S$ for all $p \in S$. We recall that an anticanonically embedded Severi–Brauer surface S is given by 27 independent quadrics. In other words, there are 27 symmetric matrices A_i such that $p \in S$ if and only if $p^T A_i p = 0$ for $1 \leq i \leq 27$. If we denote

$$G(S, F) = \left\{ g \in \mathrm{GL}_{10}(F) \mid \exists \lambda_{ij} \in F \text{ s.t. } g^T A_i g = \sum_{j=1}^{27} \lambda_{ij} A_j \right\},$$

then $\text{Aut}(S) \cong G(S, F)/Z$, where Z is a subgroup of $G(S, F)$ consisting of all scalar matrices. However, we rather work with the group $G(S, F)$, since it is conveniently given by (10×10) -matrices. Also we set

$$L(S, F) = \left\{ X \in \mathfrak{gl}_{10}(F) \mid \exists \lambda_{ij} \in F \text{ s.t. } X^T A_i + A_i X = \sum_{j=1}^{27} \lambda_{ij} A_j \right\}.$$

Lemma 3.1. *The group $G(S, \mathbb{R})$ is a Lie group and its Lie algebra is $L(S, \mathbb{R})$.*

Proof. To see that $G(S, \mathbb{R})$ is a Lie group as described in [18], we embed it into $\text{GL}_{37}(\mathbb{R})$. For $g \in G(S, \mathbb{R})$ let $\lambda_{ij} \in \mathbb{R}$ be such that $g^T A_i g = \sum_{j=1}^{27} \lambda_{ij} A_j$, then g is mapped to $\text{diag}(g, \Lambda)$ where $\Lambda = (\lambda_{ij})$ is the matrix containing λ_{ij} .

We use the characterization of the Lie algebra of $G(S, \mathbb{R})$ as the set of all $X \in M_{10}(\mathbb{R})$ such that $\exp(tX) \in G(S, \mathbb{R})$ for all $t \in \mathbb{R}$ (cf. [18]). Let X have this property. Then there are $v_{ij}(t)$ such that $(\exp(tX))^T A_i \exp(tX) = \sum_j v_{ij}(t) A_j$ for $t \in \mathbb{R}$. We differentiate this equation with respect to t and set $t = 0$. This yields $X \in L(S, \mathbb{R})$.

Suppose on the other hand that $X \in L(S, \mathbb{R})$, and let $\lambda_{ij} \in \mathbb{R}$ be such that $X^T A_i + A_i X = \sum_{j=1}^{27} \lambda_{ij} A_j$. For $s \geq 0$ and $t \in \mathbb{R}$ set

$$R_s = \sum_{r=0}^s \frac{(tX^T)^r}{r!} A_i \frac{(tX)^{s-r}}{(s-r)!}.$$

Then $(\exp tX)^T A_i (\exp tX) = \sum_{s \geq 0} R_s$. A small calculation shows

$$R_{s+1} = \frac{t}{s+1} (X^T R_s + R_s X).$$

Let $\Lambda = (\lambda_{ij})$ be the matrix containing the λ_{ij} . Then by induction we get $R_s = (1/s!) \sum_j t^s \Lambda^s(i, j) A_j$. Hence $(\exp tX)^T A_i (\exp tX) = \sum_j \exp(t\Lambda)(i, j) A_j$. It follows that X lies in the Lie algebra of $G(S, \mathbb{R})$. \square

From the discussion before the lemma it follows that $L(S, \mathbb{R})/K$, with K the subalgebra consisting of the scalar matrices in $L(S, \mathbb{R})$, is the Lie algebra of the group of automorphisms of S .

Remark 3.2. There is also an alternative way for finding the Lie algebra of $G(S, F)$ corresponding to S by understanding $G(S, F)$ as an algebraic group. For an overview of the theory of algebraic groups we refer to [12]. To describe the group $G(S, F)$ by polynomial functions, we embed it into $\text{GL}_{37}(F)$ as in the proof of Lemma 3.1. The ideal defining the image of $G(S, F)$ under this embedding is generated by two types of polynomials:

$$f_{rs}^i(\mathbf{t}) = \left(g^T A_i g - \sum_{j=1}^{27} \lambda_{i,j} A_j \right)_{rs} = \sum_{k,l=1}^{10} (A_i)_{kl} t_{kr} t_{ls} - \sum_{j=1}^{27} t_{10+i,10+j} (A_j)_{rs}$$

for $r, s = 1, \dots, 10$, $i = 1, \dots, 27$, and

$$g_{rs}(\mathbf{t}) = t_{rs} \quad \text{if } r = 1, \dots, 10, s = 11, \dots, 37 \text{ or } r = 11, \dots, 37, s = 1, \dots, 10.$$

Then the Lie algebra L consists of those $\delta = (\delta_{kl}) \in \mathfrak{gl}_{37}(F)$ that satisfy

$$\delta(f_{rs}^i) = \sum_{k,l=1}^{37} \delta_{kl} \frac{\partial}{\partial t_{kl}} f_{rs}^i \Big|_{t=e} = 0$$

(i.e., partial derivatives followed by the evaluation at the identity e) and similarly $\delta(g_{rs}) = 0$ for all relevant r, s, i . This gives us conditions for δ :

$$\bar{\delta}^T A_i + A_i \bar{\delta} = \sum_{j=1}^{27} \delta_{10+i,10+j} A_j,$$

where $\bar{\delta} = (\delta_{kl})_{k,l=1}^{10}$ is the left upper block of δ , and $\delta_{rs} = 0$ for $r = 1, \dots, 10, s = 11, \dots, 37$ or $r = 11, \dots, 37, s = 1, \dots, 10$. This is isomorphic to the algebra $L(S, F)$.

Theorem 3.3. Let $S_0 \subset \mathbb{P}^9(\mathbb{Q})$ be the Severi–Brauer surface given by the standard embedding (1). Let $L_0 = L(S_0, \mathbb{Q})$. Then L_0 is isomorphic to $\mathfrak{gl}_3(\mathbb{Q})$ and the natural 10-dimensional L_0 -module is irreducible.

Proof. We first prove the statement of the theorem over \mathbb{R} , and afterwards we revert back to \mathbb{Q} .

As above let Z denote the subgroup of $G(S_0, \mathbb{R})$ consisting of the scalar matrices. Let K denote the Lie algebra of Z . If we view K as a subalgebra of $L(S_0, \mathbb{R})$ then K coincides with the scalar matrices in $L(S_0, \mathbb{R})$. Now the Lie algebra of $G(S_0, \mathbb{R})/Z$ is isomorphic to $L(S_0, \mathbb{R})/K$. However, since $G(S_0, \mathbb{R})/Z$ is isomorphic to $\mathrm{PGL}_3(\mathbb{R})$ also the Lie algebra of $G(S_0, \mathbb{R})/Z$ is isomorphic to the Lie algebra of $\mathrm{PGL}_3(\mathbb{R})$, which is isomorphic to $\mathfrak{sl}_3(\mathbb{R})$. It follows that $L(S_0, \mathbb{R})/K$ is isomorphic to $\mathfrak{sl}_3(\mathbb{R})$. Since K is 1-dimensional we get that $L(S_0, \mathbb{R})$ is isomorphic to $\mathfrak{gl}_3(\mathbb{R})$.

Let $\{v_0, v_1, v_2\}$ be the standard basis of $V = \mathbb{R}^3$. Let $W = \mathrm{Sym}^3(V)$ with the basis

$$\{v_0^3, v_1^3, v_2^3, 3v_0^2 v_1, 3v_1^2 v_2, 3v_2^2 v_0, 3v_0 v_1^2, 3v_1 v_2^2, 3v_2 v_0^2, 6v_0 v_1 v_2\}.$$

Let $\varphi: V \rightarrow W$ be defined by $\varphi(v) = v^3$. We write the coordinates of an element of W with respect to the basis above. Then the image of the induced map $\varphi: \mathbb{P}(V) \rightarrow \mathbb{P}(W)$ is exactly S_0 , see (1).

Write $H = \mathrm{GL}_3(\mathbb{R})$. Then H acts on W by $h \cdot uvw = (hu)(hv)(hw)$, for $u, v, w \in V$. By writing the matrix of elements of H with respect to the basis above we get a representation $\rho: H \rightarrow \mathrm{GL}_{10}(\mathbb{R})$. We have $h \cdot \varphi(v) = \varphi(h \cdot v)$, and hence $\varphi(V)$ is fixed under the action of H on W . We have further $S_0 = \varphi(\mathbb{P}(V))$, therefore $\rho(H) \subseteq \mathrm{Aut}(\varphi(V)) = G(S_0, \mathbb{R})$.

Let $h = (h_{ij})_{i,j=0}^2 \in H$. Since $(\rho(h))_{ij} = h_{ij}^3$ for $i, j = 0, 1, 2$, we see that $\rho: H \rightarrow G(S_0, \mathbb{R})$ is injective. Hence by differentiating ρ we get an injective morphism of Lie algebras $d\rho: \mathfrak{gl}_3(\mathbb{R}) \rightarrow L(S_0, \mathbb{R})$. Since the dimensions of these Lie algebras are equal, this is an isomorphism. Therefore the natural $L(S_0, \mathbb{R})$ -module is isomorphic to $\text{Sym}^3(V)$ and hence irreducible.

Now we can prove the statement of the theorem for \mathbb{Q} . Let e_{ij} denote the element of $\mathfrak{gl}_3(\mathbb{R})$ that has a 1 on position (i, j) and zeros elsewhere. Since the module afforded by the representation $d\rho: \mathfrak{gl}_3(\mathbb{R}) \rightarrow \mathfrak{gl}_{10}(\mathbb{R})$ is $\text{Sym}^3(V)$, it follows then that the matrix $d\rho(e_{ij})$ has integer entries as well. Since $L_0 = d\rho(\mathfrak{gl}_3(\mathbb{R})) \cap \mathfrak{gl}_{10}(\mathbb{Q})$, we see that $d\rho(e_{ij}) \in L_0$. Hence $L_0 \cong \mathfrak{gl}_3(\mathbb{Q})$.

Finally, the natural L_0 -module is irreducible over \mathbb{Q} since it is irreducible over \mathbb{R} . \square

Now we revert back to the situation described at the beginning of the section, i.e., we let $S_0, S \subset \mathbb{P}^9(\mathbb{Q})$ be two Severi–Brauer surfaces, each given by 27 linearly independent quadrics $x^T A_i^0 x$ respectively $x^T A_i x$, $i = 1, \dots, 27$, where $x = (x_0 \dots x_9)^T$ is a column vector. The embedding of S_0 in $\mathbb{P}^9(\mathbb{Q})$ is also given by the standard embedding (1). In the sequel we work with the Lie algebras $L_0 = L(S_0, \mathbb{Q})$ and $L = L(S, \mathbb{Q})$.

Proposition 3.4. *Suppose that S_0 and S are isomorphic over \mathbb{Q} and let M be the matrix as in (2). Then the map $X \mapsto MXM^{-1}$ is a Lie algebra isomorphism from L_0 to L .*

Proof. We have that $p \in S_0$ if and only if $Mp \in S$. In other words $p^T A_i^0 p = 0$ for all i if and only if $p^T M^T A_i M p = 0$ for all i . Hence the matrices $M^T A_i M$ also describe S_0 . So there are $\eta_{ij} \in \mathbb{Q}$ such that $M^T A_i M = \sum_{j=1}^{27} \eta_{ij} A_j^0$. In the same way there are $\theta_{ij} \in \mathbb{Q}$ such that

$$(M^{-1})^T A_i^0 M^{-1} = \sum_{j=1}^{27} \theta_{ij} A_j.$$

Now let $X \in L_0$. Then we have to show that $MXM^{-1} \in L$. Using the above we calculate

$$\begin{aligned} (MXM^{-1})^T A_i + A_i M X M^{-1} &= \sum_{j=1}^{27} \eta_{ij} (M^{-1})^T (X^T A_j^0 + A_j^0 X) M^{-1} \\ &= \sum_{j,k=1}^{27} \eta_{ij} \lambda_{jk} (M^{-1})^T A_k^0 M^{-1} \\ &= \sum_{j,k,l=1}^{27} \eta_{ij} \lambda_{jk} \theta_{kl} A_l. \end{aligned}$$

The desired conclusion follows. \square

We decompose L_0 and L as direct sums of ideals as follows

$$L_0 = \langle I_{10} \rangle \oplus \{x \in L_0 \mid \text{Tr}(x) = 0\},$$

$$L = \langle I_{10} \rangle \oplus \{x \in L \mid \text{Tr}(x) = 0\}.$$

We call the ideal $\{x \in L_0 \mid \text{Tr}(x) = 0\}$ the traceless part of L_0 , and similarly for L . Then by Theorem 3.3 the traceless part of L_0 is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$. So if $L_0 \cong L$ then the traceless part of L is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$ as well.

In the next sections we describe an algorithm that given a Lie algebra N such that $N \otimes \mathbb{C} \cong \mathfrak{sl}_3(\mathbb{C})$ decides whether N is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$ and in the affirmative case explicitly constructs an isomorphism. Therefore we can decide whether L_0 and L are isomorphic. If this is not the case, we conclude that S_0 and S are not isomorphic over \mathbb{Q} , and we are done. So in the remainder of this section we assume that L_0 and L are isomorphic. Set $K = \mathfrak{sl}_3(\mathbb{Q})$, then we construct isomorphisms of K with the traceless part of L_0 and L , respectively. This yields injective homomorphisms $\varphi_0 : K \rightarrow L_0$ and $\varphi : K \rightarrow L$. We note that these maps are representations of K .

Let $H \subset K$ be a fixed Cartan subalgebra with basis h_1, h_2 which are part of a Chevalley basis of K . Let τ be a fixed automorphism of K , such that $\tau(h_1) = h_2$ and $\tau(h_2) = h_1$ (such an automorphism exists by [3, §5.11]).

Theorem 3.5. *The representation φ_0 of K is either isomorphic to φ or to $\varphi \circ \tau$. Let V_0 be the 10-dimensional K -module corresponding to φ_0 . If φ_0 is isomorphic to φ then we let V be the 10-dimensional K -module corresponding to φ , otherwise we let V be the K -module corresponding to $\varphi \circ \tau$. Let $f : V_0 \rightarrow V$ be an isomorphism of K -modules. Then f modulo scalar multiplication is also an isomorphism from S_0 to S .*

Proof. First note that the K -modules V_0 and V are irreducible. For V_0 this follows from Theorem 3.3. However, since L_0 and L are isomorphic, the same holds for V (cf. Proposition 3.4).

There are exactly two irreducible K -modules of dimension 10. We represent a weight $\lambda \in H^*$ by the tuple $(\lambda(h_1), \lambda(h_2))$. Then the two irreducible K -modules of dimension 10 have highest weights $(3, 0)$ and $(0, 3)$ respectively. By composing φ with τ we change the highest weight of the corresponding module (from $(3, 0)$ to $(0, 3)$ or vice versa). Therefore, after maybe composing φ with τ we have that the two representations have the same highest weight, and hence are isomorphic.

In order to prove the last assertion of the theorem we may work over \mathbb{C} . We consider the Lie groups H_0 and H which are generated by all exponentiations of $\varphi_0(K \otimes \mathbb{C})$, and all exponentiations of $\varphi(K \otimes \mathbb{C})$ respectively. Then H_0 and H are subgroups of $G(S_0, \mathbb{C})$ and $G(S, \mathbb{C})$ respectively (cf. Lemma 3.1). Both Lie groups are isomorphic to the Lie group generated by all exponentiations of the elements of the natural representation of $K \otimes \mathbb{C}$, which is $\text{SL}_3(\mathbb{C})$. These two isomorphisms give two representations ψ_0, ψ of $\text{SL}_3(\mathbb{C})$ in $V_0 \otimes \mathbb{C}$ and $V \otimes \mathbb{C}$, respectively, and f is an isomorphism between these two representations.

The orbits of $\text{SL}_3(\mathbb{C})$ in the representation ψ_0 are the orbits of cubic forms under linear substitution. By [5, Table 5.16], there is precisely one $\text{GL}_3(\mathbb{C})$ -orbit of dimension 3, namely the orbit of triple lines; all other orbits except the zero orbit have dimension at

least 5. The orbit of triple lines is also a single orbit under the action of $\mathrm{SL}_3(\mathbb{C})$, hence it is the only $\mathrm{SL}_3(\mathbb{C})$ -orbit of dimension 3. Its image in the projective space \mathbb{P}^9 is therefore equal to the surface S_0 . Similarly, the image of the unique 3-dimensional orbit under ψ in \mathbb{P}^9 is S . It is clear that f takes the unique 3-dimensional orbit under ψ_0 to the unique 3-dimensional orbit under ψ . \square

Remark 3.6. Let $\varphi': K \rightarrow L$ be a different embedding of K into L , and let V' be the associated K -module, which we assume to be isomorphic to V_0 . Then there is another way to see that an isomorphism $f': V_0 \rightarrow V'$ will also lead to an isomorphism $S_0 \rightarrow S$. For that set $g = \varphi'^{-1} \circ \varphi$. Then g is an automorphism of K , hence by [15, Chapter IX, Theorem 4], we infer that either g is an inner automorphism, or the composition of τ and an inner automorphism. By [3, §8.5], composing a representation with an inner automorphism does not change its highest weight. Therefore g has to be inner. First suppose that g is of the form $\exp(\mathrm{ad} z)$, where $z \in K$ is such that $\mathrm{ad} z$ is nilpotent. This implies that $\varphi(z)$ is nilpotent as well. Now by the proof of [3, Lemma 8.5.1], the map $\exp(\varphi(z))$ is a module isomorphism $V \rightarrow V'$. Furthermore, since V and V' are irreducible K -modules, this is the only isomorphism upto scalar multiples. Hence $f' = \exp(\varphi(z)) \circ f$ (upto scalar multiples). But $\exp(\varphi(z))$ lies in the automorphism group of S . It follows that f' also provides an isomorphism $S_0 \rightarrow S$. If g is a product of elements of the form $\exp(\mathrm{ad} z)$, then we reach the same conclusion.

Note that this construction generalizes to finding an isomorphism of Severi–Brauer varieties of arbitrary dimension n , since the modules involved are always symmetric powers of the natural \mathfrak{sl}_{n+1} -module, therefore irreducible.

Remark 3.7. We note that by using standard techniques from the representation theory of semisimple Lie algebras it is straightforward to construct an isomorphism between two irreducible K -modules V and W . Let λ be the highest weight of both modules, and $v_\lambda \in V$, $w_\lambda \in W$ be two corresponding highest weight vectors. Let y_1, y_2, y_3 be the negative root vectors of K , spanning the subalgebra N^- of K . Compute a set of elements u_k in the universal enveloping algebra of N^- such that the elements $u_k \cdot v_\lambda$ form a basis of V . Then a module isomorphism maps $u_k \cdot v_\lambda$ to $u_k \cdot w_\lambda$.

4. Construction of a central simple associative algebra

In this section \mathfrak{g} will be a simple Lie algebra of dimension 8 over \mathbb{Q} such that $\mathfrak{g} \otimes \mathbb{C}$ is isomorphic to $\mathfrak{sl}_3(\mathbb{C})$. The problem is to decide whether this isomorphism already exists over \mathbb{Q} , i.e., whether \mathfrak{g} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$.

As a first step towards deciding this we compute a Cartan subalgebra H of \mathfrak{g} (cf. [3, §3.2]). The problem is immediately solved if H happens to be split (i.e., $\mathrm{ad} h$ has all its eigenvalues in \mathbb{Q} for all $h \in H$). In that case we can effectively construct an isomorphism $\mathfrak{g} \rightarrow \mathfrak{sl}_3(\mathbb{Q})$, for example, by using the method of [3, §5.11]. So in the remainder of this section we suppose that H is not split.

The second step of our method is the construction of a certain rational representation of \mathfrak{g} . This works as follows. Let F denote a number field containing the eigenvalues of $\mathrm{ad} h$

for $h \in H$. Consider the Lie algebra $\mathfrak{g} \otimes F$. This Lie algebra has a split Cartan subalgebra (namely $H \otimes F$). Therefore we can find an isomorphism $\mathfrak{g} \otimes F \rightarrow \mathfrak{sl}_3(F)$. This gives us a representation $\rho' : \mathfrak{g} \otimes F \rightarrow \mathfrak{gl}(V')$, where V' is a vector space over F of dimension 3. Now we view V' as a vector space over \mathbb{Q} . More precisely, let W be the \mathbb{Q} -span of a basis of V' . Since F is a vector space over \mathbb{Q} we can form the tensor product $V = F \otimes_{\mathbb{Q}} W$, which is a vector space over \mathbb{Q} . There is a bijective \mathbb{Q} -linear map $\phi : V \rightarrow V'$ with $\phi(\alpha \otimes w) = \alpha w$. From this we get a representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, where $\rho(x) = \phi^{-1} \circ \rho'(x) \circ \phi$.

By $\rho(\mathfrak{g})^*$ we denote the associative algebra over \mathbb{Q} generated by $\rho(\mathfrak{g})$.

For an associative algebra A we let A_{Lie} be the Lie algebra associated to A (i.e., it has the same underlying vector space as A , and the Lie product is formed by $[x, y] = xy - yx$). The following lemma is immediate.

Lemma 4.1. *Set $A = \rho(\mathfrak{g})^*$. Then $\rho : \mathfrak{g} \rightarrow A_{\text{Lie}}$ is an injective homomorphism of Lie algebras.*

Lemma 4.2. *Suppose that \mathfrak{g} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$. Then $\dim_{\mathbb{Q}} \rho(\mathfrak{g})^* = 9$.*

Proof. Since \mathfrak{g} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$ we have that \mathfrak{g} has a \mathbb{Q} -basis x_1, \dots, x_8 that is a Chevalley basis (cf. [11, §25.2]). For example, we can take the image of a Chevalley basis of $\mathfrak{sl}_3(\mathbb{Q})$ under a \mathbb{Q} -isomorphism $\mathfrak{sl}_3(\mathbb{Q}) \rightarrow \mathfrak{g}$.

By [11, Theorem 27], V' has a basis B such that the matrix with respect to B of each $\rho'(x_i)$ has integer entries. In other words, there exists an $X \in \text{GL}_3(F)$ such that $X^{-1}\rho'(x_i)X$ has integer entries, and hence lies in $\mathfrak{sl}_3(\mathbb{Q})$. So I_3 (the (3×3) -identity matrix) along with the $X^{-1}\rho'(x_i)X$ form a basis of $M_3(\mathbb{Q})$ (the algebra of (3×3) -matrices over \mathbb{Q}). Hence there are $c_{ij}^k \in \mathbb{Q}$ and $e_{ij} \in \mathbb{Q}$ such that

$$X^{-1}\rho'(x_i)XX^{-1}\rho'(x_j)X = \sum_{k=1}^8 c_{ij}^k X^{-1}\rho'(x_k)X + e_{ij}I_3,$$

or

$$\rho'(x_i)\rho'(x_j) = \sum_{k=1}^8 c_{ij}^k \rho'(x_k) + e_{ij}I_3.$$

Hence the $\rho'(x_i)$ along with I_3 span a 9-dimensional associative algebra over \mathbb{Q} . We finish with the observation that the \mathbb{Q} -dimension of $\rho(\mathfrak{g})^*$ is equal to the \mathbb{Q} -dimension of $\rho'(\mathfrak{g})^*$ (since $\rho(\mathfrak{g})^* = \phi^{-1}\rho'(\mathfrak{g})^*\phi$). \square

In the third step of the method we check whether the dimension of $\rho(\mathfrak{g})^*$ is 9. If not, then \mathfrak{g} is not isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$ and we are done. If this dimension is 9 then we proceed.

Proposition 4.3. *Set $A = \rho(\mathfrak{g})^*$, and suppose that $\dim_{\mathbb{Q}}(A) = 9$. Then A is a central simple algebra. Furthermore, \mathfrak{g} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$ if and only if A is isomorphic to $M_3(\mathbb{Q})$ (the algebra of (3×3) -matrices over \mathbb{Q}).*

Proof. Set $K = \rho(\mathfrak{g})$. Then by Lemma 4.1 K is isomorphic to \mathfrak{g} . It follows that there is a direct sum decomposition $A_{\text{Lie}} = K \oplus Z$, where Z is spanned by the identity of A . Now any two-sided ideal of A is an ideal of A_{Lie} as well. However, the only ideals of A_{Lie} are 0 , Z , K and A_{Lie} . Now Z is not an ideal of A . So the only ideal of A , not equal to 0 or A , has to coincide with K . This ideal cannot be nilpotent (otherwise \mathfrak{g} would be a nilpotent Lie algebra by Engel's theorem, cf. [11]), hence the radical of A is zero. So A is the direct sum of simple ideals. But the centre of A has dimension 1 (it is equal to the centre of A_{Lie}), so there is only one simple ideal in the direct sum decomposition of A . We conclude that A is central simple.

If A is isomorphic to $M_3(\mathbb{Q})$, then A_{Lie} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q}) \oplus Z$. Hence \mathfrak{g} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$. For the other direction, if \mathfrak{g} is isomorphic to $\mathfrak{sl}_3(\mathbb{Q})$, then \mathfrak{g} contains a split Cartan subalgebra \tilde{H} . There is a basis of V with respect to which the matrix of $\rho(h)$ is diagonal for all $h \in \tilde{H}$ (cf. [3, §8.1]). Adding the identity we see that A contains a 3-dimensional split torus (i.e., a commutative diagonalisable subalgebra). Now because A is central simple, it is isomorphic to $M_r(D)$, where D is a division ring over \mathbb{Q} . But such an algebra only contains a 3-dimensional split torus if $r = 3$ and $D = \mathbb{Q}$. \square

Remark 4.4. A question arises: is an embedding $\mathfrak{g} \rightarrow A_{\text{Lie}}$ unique? This is answered in [15, Chapter 10, Theorem 10], namely, there are either precisely two such embeddings or there are none.

Now let A be as in the lemma, and suppose that A is isomorphic to $M_3(\mathbb{Q})$. Let $\tau : M_3(\mathbb{Q}) \rightarrow A$ be an isomorphism. Then we take the associated Lie algebras of these associative algebras, and restrict τ to the subalgebra $\mathfrak{sl}_3(\mathbb{Q}) \subset (M_3(\mathbb{Q}))_{\text{Lie}}$. We obtain an embedding $\tau : \mathfrak{sl}_3(\mathbb{Q}) \rightarrow A_{\text{Lie}}$. The image of this map is the unique semisimple subalgebra of A_{Lie} , i.e., $\rho(\mathfrak{g})$. Hence after composing with ρ^{-1} we obtain an isomorphism $\mathfrak{sl}_3(\mathbb{Q}) \rightarrow \mathfrak{g}$.

5. Associative central simple algebra

In this section we start with a central simple algebra A of degree 3 and we want to decide whether A is isomorphic to $M_3(\mathbb{Q})$. In the affirmative case we also want to construct an isomorphism.

Since the degree of the algebra A is prime, by Wedderburn's structure theorem there are only two possibilities: either $A \cong M_3(\mathbb{Q})$ or A is a division algebra. So if we by chance find a zero divisor in A , then we can already conclude, that $A \cong M_3(\mathbb{Q})$. Using the zero divisor we can even explicitly construct an isomorphism.

Let $a \in A$ be a zero divisor, so $A \cong M_3(\mathbb{Q})$. We can find a 3-dimensional left ideal. Namely, the vector space endomorphism ρ_a of $A : x \mapsto xa$ has a nontrivial kernel (because a is a zero divisor) and a nontrivial image (because $1.a \neq 0$). Both $\text{Ker } \rho_a$ and $\text{Im } \rho_a$ are left ideals in A . Since any minimal left ideal of $M_3(\mathbb{Q})$ is of dimension 3 and any left ideal is a direct sum of minimal left ideals, we have either $\dim(\text{Ker } \rho_a) = 3$ or $\dim(\text{Im } \rho_a) = 3$.

Let \mathcal{L} be a 3-dimensional left ideal in A . Let $B = (b_1 \ b_2 \ b_3)^T$ be the column vector containing a basis of \mathcal{L} . Let $\varphi : A \rightarrow M_3(\mathbb{Q})$ be the map that assigns to $x \in A$ the transpose of the matrix of its left action on \mathcal{L} w.r.t. B , so $\varphi(x) = X$, if $X^T B = (xb_1 \ xb_2 \ xb_3)^T$.

Theorem 5.1. φ is an isomorphism of algebras.

Proof. Elementary computations show that φ is a homomorphism of algebras. Further φ is a bijection, since otherwise $\text{Ker } \varphi \neq 0$ would be a nontrivial ideal in A . \square

Remark 5.2. The problematic part of the foregoing construction is the assumption that we have a zero divisor in the algebra and so we can construct a 3-dimensional left ideal. There is an ongoing research on this topic ([25], the case of algebras of degree 2 was solved in [14]). First, a maximal order in the algebra is constructed so the structure constants are integral (cf. [13]). Afterwards, the basis of the order is changed to reduce the size of the structure constants substantially. This makes finding a zero divisor possible. Here we follow another approach, which is similar to the one in [1]. However, the method of [1] does not provide an explicit isomorphism. We also note that by finding a maximal order we can decide whether a given central simple algebra is isomorphic to the full matrix algebra (see [22]). However, this does not yield an explicit isomorphism.

For finding an isomorphism of the algebra A and $M_3(\mathbb{Q})$ we first express A as a cyclic algebra of degree 3, i.e., we find two elements a, b of A so that the algebra generated by them satisfies the following

Definition 5.3. A is a *cyclic algebra* of degree 3 if there are elements $a, b \in A$ so that

$$A = \langle 1, a, a^2, b, ab, a^2b, b^2, ab^2, a^2b^2 \rangle_{\mathbb{Q}},$$

and the multiplication rules satisfy the following conditions:

- (i) $E = \langle 1, a, a^2 \rangle_{\mathbb{Q}}$ is a Galois extension of \mathbb{Q} with $\text{Gal}(E|\mathbb{Q}) = \langle \sigma \rangle$,
- (ii) $ba = \sigma(a)b$,
- (iii) $b^3 = \beta 1$, where $\beta \in \mathbb{Q}^*$.

In this case we write $A = (E, \sigma, \beta)$.

Any cyclic algebra is central simple. For algebras of degree 3, the reverse implication also holds: any central division algebra of degree 3 is cyclic (a result due to Wedderburn) and likewise the split algebra $M_3(\mathbb{Q})$ (note that $(E, \sigma, 1)$ is split for any E). Hence in any case there exists an isomorphism of A and a cyclic algebra.

In the construction of an isomorphism the most difficult step is to find a cyclic element, i.e., an element $a \in A$ such that the minimal polynomial $m_a(\lambda) \in \mathbb{Q}[\lambda]$ is irreducible of degree 3 and the splitting field of m_a has dimension 3 over \mathbb{Q} . For finding a cyclic element we follow [17, Lemma 2.9.8 and Theorem 2.9.17, pp. 68–70]. Although the construction is originally designed for division algebras, it can be partially carried out also in the case $A \cong M_3(\mathbb{Q})$. In fact, the only complication which could arise is that we hit a noninvertible element and so can not continue with the original computation. But in such case we have found a zero divisor and we can construct the isomorphism $A \xrightarrow{\sim} M_3(\mathbb{Q})$ as in Theorem 5.1.

We will need the following properties of the elements in the matrix algebra.

Lemma 5.4. *For a field F let $A \cong M_3(F)$ and let $a \in A$, a not central. Let the minimal polynomial $m_a(\lambda) \in F[\lambda]$ of a be irreducible over F . Then $\deg m_a(\lambda) = 3$ and every $b \in A$, such that $m_a(b) = 0$ is a conjugate of a .*

Proof. $a \notin Z(A)$ implies $\deg m_a(\lambda) > 1$. The case $\deg m_a(\lambda) = 2$ is not possible. For, let $\deg m_a(\lambda) = 2$, $m_a(\lambda)$ irreducible. Then the characteristic polynomial $\chi_a(\lambda)$ of a is $\chi_a(\lambda) = m_a(\lambda)l(\lambda)$ with $l(\lambda)$ linear. The factor $l(\lambda)$ of the characteristic polynomial of a is irreducible, therefore it divides the minimal polynomial $m_a(\lambda)$, a contradiction. Since $m_a(\lambda)$ is irreducible over F and $m_a(b) = 0$, it is also the minimal polynomial of b . Then $I\lambda - a$ and $I\lambda - b$ have the same invariant factors and hence a and b are conjugate (cf. [24]). \square

Now we want to construct a cyclic element $a \in A$. We start by picking a noncentral element $x \in A$. If the minimal polynomial $m_x(\lambda)$ of x is reducible over \mathbb{Q} , we can construct a zero divisor and hence an isomorphism $A \xrightarrow{\sim} M_3(\mathbb{Q})$. Indeed, let $m_x(\lambda) = m_1(\lambda)m_2(\lambda)$, then $m_1(x), m_2(x) \neq 0$ are zero divisors. On the other hand, if $m_x(\lambda)$ is irreducible, then $\deg m_x(\lambda) = 3$. For $A \cong M_3(\mathbb{Q})$ it follows from Lemma 5.4. If A is a central division algebra, then $m_x(\lambda)$ is the minimal polynomial of x also after extending the field of coefficients. If F is a splitting field of A , so $A \otimes_{\mathbb{Q}} F \cong M_3(F)$, then again by Lemma 5.4 $\deg(m_x) = 3$. Therefore in our construction we can assume that the minimal polynomial of a randomly chosen element $x \in A$ is irreducible and has degree 3.

Recall that $[a, b] = ab - ba$.

Lemma 5.5. *Let A be a central simple algebra over \mathbb{Q} . Let $x \in A$ be a noncyclic element such that its minimal polynomial $m_x(\lambda)$ is cubic and irreducible. Then there exists $t \in A$ such that $[[t, x], x] \neq 0$.*

Proof. Let $[[t, x], x] = 0$ for all $t \in A$. For the inner derivation $i_x: y \mapsto [y, x]$ we have $i_x^2 = 0$, where $i_x^2: y \mapsto [[y, x], x] = yx^2 - 2xyx + x^2y$. But the vector space endomorphisms $y \mapsto yx^2$, $y \mapsto xyx$ and $y \mapsto x^2y$ are linearly independent (see [16, the proof of Theorem 4.6, p. 218]), a contradiction. \square

For given x , $[[t, x], x] \neq 0$ is a linear condition on t , so it is easy to find a desired $t \in A$.

Lemma 5.6. *Let $x, t \in A$ be as in Lemma 5.5 and let $[t, x]$ be invertible. Let $y = [t, x]x[t, x]^{-1}$ and $z = [x, y]$. Then $z \neq 0$, $z \notin \mathbb{Q}$, $z^3 \in \mathbb{Q}$.*

Proof. The assertion follows from [17, §2.9, Lemma 2.9.8]. There it is proven for central division algebras, but the proof for the case $A \cong M_3(\mathbb{Q})$ is almost a word-by-word copy of the original one. The proof is technical and therefore we omit it here. \square

Theorem 5.7. *Let A be a central simple algebra over \mathbb{Q} of degree 3. We can either find a zero divisor in A or we can find elements $a, b \in A$ generating the algebra A as a cyclic algebra as described in Definition 5.3.*

Proof. Let $x \in A$ be an arbitrary element. Then either x fulfills the assumptions of Lemma 5.5, or x is a zero divisor and we are done, or x is a cyclic element. In this case (x cyclic) we set $a = x$ and denote $E = \langle 1, a, a^2 \rangle_{\mathbb{Q}}$, the maximal subfield generated by a . Let σ denote a generator of $\text{Gal}(E|\mathbb{Q})$. By factoring the minimal polynomial $m_a(\lambda)$ over E we find $\sigma(a)$ and afterwards an element $b \in A$ s.t. $bab^{-1} = \sigma(a)$, i.e., $b \notin E$. By Lemma 5.4 such b exists if $A \cong M_3(\mathbb{Q})$, for A a division algebra we refer to [17]. We claim that b satisfies also the rule (iii) of Definition 5.3. Indeed, by multiple conjugation we get $b^3ab^{-3} = a$, so $b^3 \in Z_A(E) = E$. If $b^3 \notin \mathbb{Q}$, then b^3 would generate E . But in such case $b^3b = bb^3$ would imply $b \in Z_A(E) = E$, a contradiction.

Now we will construct elements $a, b \in A$ and prove that they match Definition 5.3 in case x is noncyclic and not a zero divisor. Note, that we may assume that the minimal polynomials of a and b are irreducible, otherwise we would have found a zero divisor. First we construct z as in Lemma 5.6 (unless $[t, x]$ is a zero divisor, in which case we are done) and denote it by z_x . We set $b = z_x$. Then we carry out the construction of the Lemma 5.6 again, using the element b as x to get the corresponding z which we denote here z_b . Set $a = bz_b$. By [17, proof of Theorem 2.9.17, p. 69] a is a cyclic element not in \mathbb{Q} . If we define σ on E : $\sigma(a) = bab^{-1}$, a technical computation shows that $\sigma(a)a = a\sigma(a)$, i.e., $\sigma(a) \in E$, so σ is a well-defined automorphism of E . The elements a, b satisfy the properties of Definition 5.3 and hence generate a cyclic algebra (E, σ, β) . \square

If we were not lucky enough so far to hit a zero divisor in the algebra A and to use it for constructing an isomorphism $A \xrightarrow{\sim} M_3(\mathbb{Q})$ as described in the beginning of the section, then we now have an isomorphism of A and a cyclic algebra (E, σ, β) . For any cyclic algebra of degree 3 we can already decide whether it is split or a division algebra over \mathbb{Q} , as can be seen in the following

Theorem 5.8. $(E, \sigma, \beta) \cong M_3(\mathbb{Q})$ if and only if there exists $x \in E$ such that

$$x\sigma(x)\sigma^2(x) = \frac{1}{\beta}. \quad (3)$$

Proof. The algebra $(E, \sigma, \beta) \cong M_3(\mathbb{Q})$ exactly if we can find a 3-dimensional left ideal. Let \mathcal{L} be such an ideal in (E, σ, β) and let $a'_0 1 + a'_1 b + a'_2 b^2 \in \mathcal{L}$, $a'_i \in E$. At least one of a'_i 's is nonzero, let it be a'_0 (the other cases are treated in the same way). Then $1 + a_1 b + a_2 b^2 \in \mathcal{L}$. After multiplying from the left by elements from $E = \langle 1, a, a^2 \rangle_{\mathbb{Q}}$ we see that

$$\mathcal{L} = \langle 1 + a_1 b + a_2 b^2, a(1 + a_1 b + a_2 b^2), a^2(1 + a_1 b + a_2 b^2) \rangle_{\mathbb{Q}}.$$

Since \mathcal{L} is a left ideal, then also $b(1 + a_1 b + a_2 b^2) = \beta\sigma(a_2) + b + \sigma(a_1)b^2 \in \mathcal{L}$. This holds, if $\beta\sigma(a_2) : 1 : \sigma(a_1) = 1 : a_1 : a_2$, which can be satisfied if and only if a_1 is a solution to the norm equation (3). \square

In our algorithm we use an existing routine in Magma to test whether the norm equation (3) is solvable. In the affirmative case we use a solution for constructing an isomorphism of (E, σ, β) and $M_3(\mathbb{Q})$. Namely, if $x \in E$ is a solution, then

$$\mathcal{L} = \langle 1 + xb + x\sigma(x)b^2, a(1 + xb + x\sigma(x)b^2), a^2(1 + xb + x\sigma(x)b^2) \rangle_{\mathbb{Q}}$$

is a 3-dimensional left ideal and we can find an isomorphism of A and $M_3(\mathbb{Q})$ as in Theorem 5.1.

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