JOHANNES KEPLER
UNIVERSITÄT LINZ
Netzwerk für Forschung, Lehre und Praxis

Symbolic Methods for Factoring Linear Differential Operators

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Betreuung:

O. Univ.Prof. Dipl.-Ing. Dr. Franz Winkler

Mitbetreuung:

Dr. Günter Landsmann

Begutachtung:

Erster Begutachter: O. Univ.Prof. Dipl.-Ing. Dr. Franz Winkler
Zweiter Begutachter: O. Univ.Prof Dipl.-Math. Dr. Jochen Pfalzgraf

Eingereicht von:

Glauco Alfredo López Díaz, M.Sc.-Math.

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Johannes Kepler Universität
A-4040 Linz · Altenbergerstraße 69 · Internet: http://www.uni-linz.ac.at · DVR 0093696

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Abstract

A survey of symbolic methods for factoring linear differential operators is given. Starting from basic notions – ring of operators, differential Galois theory – methods for finding rational and exponential solutions that can provide first order right-hand factors are considered. Subsequently several known algorithms for factorization are presented. These include Singer's eigenring factorization algorithm, factorization via Newton polygons, van Hoeij's methods for local factorization, and an adapted version of Padé approximation.

In addition a procedure based on pure algebraic methods for factoring second order linear partial differential operators is developed. Splitting an operator of this kind reduces to solving a system of linear algebraic equations. Those solutions which satisfy a certain differential condition, immediately produce linear factors of the operator. The method applies also to operators of third order, thereby resulting in a more complicated system of equations. In contrast to the second order case, differential equations must also be solved, which, in particular cases, are simplified with the aid of characteristic sets.

Finally, complete decomposition into linear factors of ordinary differential operators of arbitrary order is discussed. A splitting formula is developed, provided that a linear basis of solutions is available. This theoretical representation is valuable in understanding the nature of the classical Beke algorithm and its variants like the algorithm LODEF by Schwarz and the Beke-Bronstein algorithm.

Zusammenfassung

Es wird ein Überblick über symbolische Methoden zur Faktorisierung linearer Differentialoperatoren gegeben. Beginnend mit grundlegenden Begriffen wie Operatorring oder differentielle Galoistheorie, diskutiert der Autor Methoden zum Auffinden rationaler wie auch exponentieller Lösungen, mit deren Hilfe rechte Faktoren von Operatoren gefunden werden können. Im Folgenden werden verschiedene Faktorisierungsalgorithmen vorgestellt, darunter Singers Eigenring-Faktorisierungsalgorithmus, Faktorisierung mit Hilfe von Newton Polygonen, van Hoeijs Methode zur lokale Faktorisierung und eine adaptierte Version der Padé Approximation.

Darüberhinaus entwickelt der Autor eine rein algebraische Methode zur Faktorisierung von linearen partiellen Differentialoperatoren. Das Zerlegen so eines Operators reduziert sich auf das Auffinden der Lösungen eines linearen algebraischen Gleichungssystems. Diejenigen Lösungen dieses Systems, welche eine bestimmte differentielle Bedingung erfüllen, erzeugen direkt lineare Faktoren des Operators. Dieselbe Methode ist auch auf Operatoren dritter Ordnung anwendbar, wobei ein komplexeres Gleichungssystem auftritt. Im Gegensatz zum Fall 2. Ordnung müssen hier auch Differentialgleichungen gelöst werden, welche sich manchmal mit Hilfe charakteristischer Mengen vereinfachen lassen.

Zuletzt wird die vollständige Zerlegung gewöhnlicher Differentialoperatoren von beliebiger Ordnung in lineare Faktoren behandelt. Unter Zugrundelegung einer linearen Basis des Lösungsraums wird eine Zerlegungsformel entwickelt. Diese theoretische Darstellung erweist sich als hilfreich zum Verständnis des klassischen Beke-Algorithmus und seiner Varianten - wie Beke-Bronstein-Algorithmus oder Schwarz' LODEF-Algorithmus.

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1. INTRODUCTION

1.1 Historical Motivation

Let (k, δ) be a differential field of characteristic 0 with algebraically closed field of constants $\mathcal{C}$. We will write $y^{(n)}$ instead of $\delta^n(y)$ and $y', y'', \ldots$ for $\delta(y), \delta^2(y), \ldots$. Let $\mathcal{D} = k[\partial]$ be the ring of linear differential operators over k, that is, the non-commutative polynomial ring in the variable ∂, where

$$\partial a = a\partial + a' \text{ for all } a \in k.$$

Any linear differential operator $L \in k[\partial]$ of the form

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0$$

defines an order n linear homogeneous scalar differential equation $L(y) = 0$ by

$$y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1 y' + a_0 y = 0.$$

Factorizing a linear differential operator L into a product $L = QR$, simplifies the computation of solutions as a solution of $R(y) = 0$ is a solution of $L(y) = 0$ as well. Moreover, we can find linearly independent solutions. As an example consider the following linear ordinary differential operator:

$$\partial^2 + (x-1)\partial - x$$

whose factorization is

$$(\partial + x)(\partial - 1).$$

Its scalar equation can be written as

$$y'' + (x-1)y' - xy = \left(\frac{d}{dx} + x\right)(y' - y) = 0$$

which has linearly independent solutions

$$y_1 = e^x \text{ and } y_2 = e^x \int e^{-\frac{x^2}{2} - x} dx.$$

In particular, an operator L is said to be reducible if there exists operators L_1 and L_2 of lower order such that $L = L_2 L_1$, in this case we say that L_1 is a right factor and L_2 is a left factor of L. If an operator is not reducible then it is called irreducible.

From Landau [1902], we know that any two decomposition of an operator L into irreducible components have the same number of factors and their orders are the same up to permutations. As an example consider the derivative $\delta = d/dx$ in the field k then the following linear differential operator

$$\partial^2 = \partial\partial = \left(\partial + \frac{1}{x-c}\right)\left(\partial - \frac{1}{x-c}\right),$$

with $c \in \mathcal{C}$ has essentially two different factorizations.

Even when the intention of the this part of the thesis is to study the existent algorithms for factoring linear ordinary differential operators, in section 2.3 we will discuss rational and exponential solutions of linear ordinary homogeneous differential equations. However, we would like to mention that there are several algorithms for solving linear differential equation in particular we give a short survey about them:

1. Rational solutions:

   Rational solutions are elements of k. To decompose the operators of this class, Liouville [1833a,b] already gave an algorithm, but only when k is a rational function field over the constants. More general versions have been presented by Singer [1991] and Bronstein [1992a]. To solve scalar equations, there are implementations by Abramov and Bronstein in MAPLE and in the BERNINA package. To solve matrix equations, there is an implementation by M. Barkatou in the ISOLDE package. Both cases are restricted to the field $\overline{\mathbb{Q}}(x)$ of rational functions in x with coefficients in the algebraic closure of $\mathbb{Q}$.

2. Algebraic solutions:

   Algebraic solutions are solutions lying in an algebraic extension of k; i.e., they satisfy an irreducible polynomial over k. An example for this class is $\sqrt[3]{1-\sqrt{x}}$.

   Many renowned mathematicians like Pépin [1881], Fuchs [1875, 1878], Klein and Jordan [1878] were searching for an algorithm for algebraic solutions. Today there exists an algorithm by Singer [1979] with some improvements in Singer and Ulmer [1993], but it is far from being satisfactory because it is very expensive; in this case one needs to substitute a minimal polynomial decomposed into invariants in the differential equation. Another method from Fakler [1997] combines Liouvillian solutions with the algebraic case of Risch's algorithm (see Bronstein [1997]), it has not been implemented.

3. Liouvillian solutions:

   Liouvillian solutions are solutions generated by repeatedly adjoining algebraic numbers, integrals or exponentials of integrals. An example of such a generation is

   $$x \xrightarrow{\sqrt{x}} \sqrt{x} \xrightarrow{e^{\int}} \exp\left[\int \sqrt{x}\right].$$

   In Singer [1981] it was shown that given a homogeneous linear differential equation $L(y) = 0$ with coefficients in F, a finite algebraic extension of $\mathbb{Q}(x)$, one can find in a finite number of steps, a basis for the vector space of Liouvillian solutions of $L(y) = 0$.

   For order two equations the method is given by Kovacic [1986], with small improvements by Weil [1994]. For order three, the methods are given by Singer and Ulmer [1993].

   Although the problem of finding all the Liouvillian solutions of a homogeneous linear differential equation is decidable in theory for any order Singer [1991], the published decision procedure is not consider a practical algorithm and has not been implemented.

4. Exponential solutions:

   An exponential solution of the equation $L(y) = 0$ is a solution whose logarithmic derivative $\frac{y'}{y}$ lies in k.

   Exponential functions form the most important subclass of Liouvillian functions. Procedures providing algorithms that produce exponential solutions lie at the basis of all known algorithms for finding Liouvillian solutions. Fortunately, there are algorithms for exponential solutions. The very first one is from Beke [1894]. For the formal case, van Hoeij [1997a] has implemented classical methods. Factorization over $\overline{\mathbb{Q}}(z)$ is based on the formal case.

   Dividing functions into one of these classes is not always unique; e.g., for the function $y = \sqrt{x}$, it is possible to attach it either to the algebraic functions or to the exponential, since $\frac{y'}{y} = \frac{1}{2x}$, and $\frac{1}{2x} \in k$. Furthermore, it can be hard to decide for a given function whether it is Liouvillian and how one could find the simplest construction. A combination of the above methods is used in the implementation by van Hoeij in the MAPLE computer algebra system.

For factorization of linear differential operators which constitutes the main subject of this thesis, we have the following known methods:

1. Beke's method: In 1894 Beke gave a method for factorization of linear differential operators in the ring $\overline{\mathbb{Q}}(x)[\partial]$. Previous implementations for factorization in $\overline{k}(x)[\partial]$ are based on his method. For example, the factorizer in the Kovacic algorithm (Kovacic [1986]) is based on Beke's method. To find a factor via Beke's method one must first compute another operator (the second exterior power) and then compute a first order right-hand factor. Construct an auxiliary operator $\widetilde{L}$ whose associated Riccati equations have among their solutions all possible coefficients b_i of factors

$$M = \partial^m + b_{m-1}\partial^{m-1} + \cdots + b_1\partial + b_0\partial^0$$

of L. From $\widetilde{L}$ one can bound the degrees of the numerators and denominators of these coefficients.

An implementation of Beke's method has been accomplished by Bronstein [1992b] in the Axiom system. Schwarz [1989] has implemented the full algorithm for equations of small order in Scratchpad II. Simplifications of the the Beke algorithm as well as a detailed complexity analysis can be found in Grigoriev [1990a]. There is an algorithm for determining the reducibility of a differential system in Grigoriev [1990b]. A method to enumerate all factors of a differential operator is given in Tsarev [1996].

2. The eigenring method: The eigenring of a differential equation $L(y) = 0$ is the finite dimensional $\mathcal{C}$-algebra of all the endomorphisms of the equation, where $\mathcal{C}$ is the subfield of constants of k. This eigenring is the set of all rational solutions of other differential equations associated to L. If this eigenring is not too trivial, then some factorizations of L can be deduced from it. The method was introduced by Singer [1996] and has been improved by van Hoeij [1997b].

3. Van Hoiej's methods for factoring differential operators are not based on Beke's algorithm. He uses algorithms to find local factorizations (i.e., factors with coefficients in $\overline{k}(x)$) and applies an adapted version of Padé approximation to produce a global factorization.

In order to do this, one should make a good choice of a singular point of the operator L and a formal local right-hand factor of degree 1 at this point. After a translation of the variable ($x \mapsto x + p$ or $x \mapsto x^{-1}$) and a shift $\partial \mapsto \partial + e$ with $e \in \overline{k}(x)$, the operator L has a right-hand factor of the form $\partial - \frac{y'}{y}$ with an explicit $y \in \overline{k}[[x]]$. Now one tries to find out whether $\frac{y'}{y}$ belongs to $\overline{k}(x)$. Equivalently, one tries to find a linear relation between y and y' over $\overline{k}[x]$. This is carried out by a Padé approximation. The method extends to finding right-hand factors of higher degree and yields in that case a generalization of the Padé approximation.

This local-to-global approach has been implemented in Maple V.5 computer algebra system.

4. Full factorizations: The structure of all possible factorizations of an ordinary differential operator is known due to a fundamental theorem of Loewy [1906]: An ordinary operator has unique factorization into completely reducible factors, i.e., in operators that have enough right factors. Recent work of Tsarev [1996] combined the local formal factorization of the previous case with the classical work of Beke [1894].

In general, the main disadvantage of all factorization algorithms is their tremendous complexity, especially if the order if the given operator is higher than two.

Much less is known about factorization of linear partial differential operators. In the 19th century, a vast interest in finding solutions of non-linear partial differential equations resulted in the development of the methods of Lagrange, Monge, Boole, and Ampere. In particular, Darboux [1870] generalized the method of Monge (known as the method of intermediate integrals) to obtain the most powerful method in those days for explicitly integrating partial differential equations.

In Anderson and Kamran [1997], Juras [1995], and Zhiber, Sokolov, and Startsev [1995], the Darboux method was put in a more precise and efficient (although not completely algorithmic) form. For the case of a single second order non-linear partial differential equation of the form

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1.1.1}$$

the idea was to linearize it. Using the substitution

$$u(x,y) \to u(x,y) + \epsilon v(x,y)$$

and canceling the terms proportional to ϵ^n, $n > 1$, we obtain the linear partial differential equation

$$v_{xy} = Av_x + Bv_y + Cv \tag{1.1.2}$$

with coefficients A, B, and C depending on x, y, u, u_x, u_y. Studying equations of type (1.1.2), Laplace invented a method for their transformation that is sometimes called the Laplace cascade method. First, the corresponding linear partial differential operator satisfies the condition

$$\begin{gathered} L = \partial_x\partial_y - A\partial_x - B\partial_y - C = \\ (\partial_x - B)(\partial_y - D) + H = (\partial_y - A)(\partial_x - B) + K, \end{gathered} \tag{1.1.3}$$

where

$$H = \partial_x A - AB - C \text{ and } K = \partial_y B - AB - C$$

are the Laplace invariants of Equation (1.1.2). Therefore, if either $H = 0$ or $K = 0$, the second order linear partial differential operator L is factorable, and the solutions of Equation (1.1.2) can be found through quadratures. If both H and K vanish, L is a left least common multiple of the two first order linear partial differential operators. If both H and K are nonzero, the two Laplace transformations $L \to L_1$ and $L \to L_{-1}$ can be applied using the substitutions

$$v_1 = (\partial_y - A)v, \;\; v_{-1} = (\partial_y - B)v. \tag{1.1.4}$$

These (invertible) transformations result in two new second order linear partial differential operators L_1 and L_{-1} of the same form as (1.1.2) with different coefficients if and only if $H \neq 0$ and $K \neq 0$. In the general case, we obtain the two infinite sequences

$$L \to L_1 \to L_2 \to \cdots$$

$$L \to L_{-1} \to L_{-2} \to \cdots$$

If one of these sequences is finite (i.e., the corresponding Laplace invariant vanishes at some step, and the Laplace transformation cannot be applied further), then the final linear partial differential operator L_i is trivially factorable.

We can consider initial Equation (1.1.1) and calculate all Laplace invariants and Laplace transformations (which means that we express all the mixed derivatives of u in terms of x, y, u, and the non-mixed derivatives $u_{x\ldots x}$ and $u_{y,\ldots,y}$).

Theorem 1. *A second order scalar hyperbolic partial differential equation of the form (1.1.1) is Darboux integrable if and only if both its Laplace sequences are finite.*

In Juras [1995] and Anderson and Kamran [1997], this method was also generalized to the case of a general second order non-linear hyperbolic partial differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

The original "Darboux method" (as Darboux stated in Darboux [1870]) is extendable in principle to equations of all orders in an arbitrary number of variables, even to systems of equations; however, in Darboux [1870] and subsequent papers by Goursat, Gau, Gosse, Vessiot, et al., the detailed calculations were performed only for a single second order equation with one dependent and two independent variables.

On the other hand, Blumberg [1912] and Miller [1932] have discussed the necessity of a suitable generalization of the concept of completely reducible operators to partial operators, and they have illustrated this problem with few typical examples. In particular, in Blumberg [1912] is given an example of a third-order operator which has two different factorizations into completely reducible factors. With this it is shown that the result of Loewy about the uniqueness of the factorization into completely reducible factors, is not true for partial differential operators.

Nowadays, due to the growing interest in Computer Algebra and the use of the Computer Algebra Systems, one tendency is treat factoring as finding superideals of a left ideal in the ring of linear partial differential operators rather than factoring a single linear differential partial operator, as done by Tsarev [2000] and Li, Schwarz, and Tsarev [2003]. In Tsarev [2000] a concept of factorization is developed which makes some characteristic factors to be uniquely defined similar to the case of ordinary operators. In Li et al. [2003] the factorization of systems of linear partial differential operators with a finite-dimensional (over the subring of constants) space of solution is studied, then the linear differential subvarieties are viewed as the factors of the input systems.

Another tendency is to try to imitate the procedures and use the techniques for factoring polynomials, as done by Grigoriev and Schwarz [2004] with their algorithm "Hensel Decent" for factoring lineal partial differential operators of arbitrary order. They have named it in that way because it is close in nature to the well-known Hensel lifting.

Grigoriev and Schwarz consider the homogeneous part of a differential operator, they define the symbol of an operator as the homogeneous polynomial with the same coefficients as the homogeneous part and the same powers as the corresponding derivatives. They define an operator to be separable if its symbol is separable, i.e., if all the roots of the symbol are distinct in an algebraically closed field extension of the field of coefficients. If the operator is separable then to find its possible factorizations reduces to polynomial factorization in the field of coefficients, rational operations in that field and taking derivatives.

We present a naive approach for factoring second order linear differential operators into linear factors in the ring $k(x, y)[\partial_x, \partial_y]$, the ring of linear partial differential operators in the indeterminates ∂_x, ∂_y, with coefficients in the field of rational functions over the field k of characteristic zero, i.e., an operator of the form

$$L := \partial_x^2 + E\partial_x\partial_y + D\partial_y^2 + C\partial_x + B\partial_y + A$$

with coefficients $A, B, C, D, E \in k(x, y)$.

In the very general case we do not need to solve any differential equation, we only need to find a square root and solving a system of linear equations plus a test equation (a first order linear partial differential equation), if we are lucky and there exists the square root, the system has a unique solution and afterwards if the test equation is satisfied, we can get a factorization in linear factors.

Our result improves the theorem 3.1 (Miller [1932]) of Grigoriev and Schwarz [2004], because we do not only propose a possible right factor of a partial differential operator of second order, but rather we find the factorization at once when it exists, with this we avoid the division in each particular case; moreover, we do not make case distinction.

1.2 Outline of the Thesis

The thesis is organized in the following way:

- Chapter 2 is the motivation for the study of the symbolic treatment for the factorization of linear differential operators. In Section 2.1 we present the basic definitions related with linear differential operators. In Section 2.2 we introduce the main contribution of the thesis, a naive approach for factoring second order linear partial differential operators in the ring $k(x, y)[\partial_x, \partial_y]$, the algorithm *gl*1 derived from it and one example of the procedure.

  In Section 2.3 we extend the procedure to linear partial differential operators of third order. We propose answers for the two possible factorizations, reducing the problem to solving a system of algebraic and differential equations, which can be triangularized by characteristic sets. In Section 2.4 we give an outline of the recent algorithm called "Hensel Decent" due to Grigoriev and Schwarz [2004].

- In Chapter 3 we start the study of the factorization algorithms for linear ordinary differential operators. Section 3.1 contains the basic facts about Galois theory of linear differential equations, starting with differential field extensions and a factorization formula in the settings of the ring of linear differential operators, provided available a fundamental set of solutions.

  Section 3.2 deals with the generalization of the Frobenius method for solving second order ordinary differential equations, using the natural embedding $k(x) \hookrightarrow k((x))$, partial fraction decomposition and indicial equations.

  Section 3.3 is devoted to finding exponential solutions of linear homogeneous differential equations in a particular case, i.e to finding rational solutions of the Riccati equation associated to the given linear equation with rational function coefficients and solutions of the same kind. We study the RiccatiRational algorithm due to Schwarz [1994], which searches for bounds on the coefficients of a possible solution and reduces to solving a linear system. If this system is feasible we obtain a rational solution of the associated Riccati equation and at once a right-hand factor of the operator corresponding to the original equation.

  In Section 3.4, the core of the thesis, we study Beke's algorithms for finding right-hand factors of linear differential operators. The main idea of Beke's algorithm is to decide in finitely many steps if a differential operator is reducible or not, and - in the first case - to construct a non-trivial right-hand factor. For this one must first compute another operator, the second exterior power. The main obstacle for using this approach is its tremendous complexity.

  In Section 3.5 we present the algorithm $LODEF$ of Schwarz [1989], which is implemented in Scratchpat II. In $LODEF$, which is the first algorithm that appeared after Beke's algorithm, Schwarz modifies Beke's algorithm making it recursively reducing the order of possible right factors. He estimates degree bounds for the coefficients of right factors and computes the size of rational solutions of certain differential equations. Later Schwarz developed the RiccatiRational algorithm thereby specifying the last step of the Beke algorithm.

  In Section 3.6 we introduce the efficient algorithm due to Bronstein [1994], for computing the associated equations appearing in Beke's method.

- Chapter 4 gives an outline of more advanced methods beyond Beke's approach. Section 4.1 deals with Singer's eigenring factorization algorithm, an adaptation of the Berlekamp algorithm to the ring of operators. Section 4.2 is devoted to a geometric factorization method using a generalization of Newton polygons. Section 4.3 contains van Hoeij's techniques for local factorization in $k((x))[\partial]$ (i.e., the ring of linear differential operators with coefficients in the field of formal Laurent series in the indeterminate x over k) using new notions of Newton polynomials and coprime factorization.

  Finally, in Section 4.4, we present van Hoeij's method for local factorization over $k(x)$ using a generalization of Padé approximation.

Part I

FACTORIZATION OF LINEAR PARTIAL DIFFERENTIAL OPERATORS

2. FACTORIZATION OF LINEAR PARTIAL DIFFERENTIAL OPERATORS

In this chapter we motivate the study of symbolic factorization of linear differential operators. We give a naive approach for factoring second order linear differential operators from $k(x, y)[\partial_x, \partial_y]$ into linear factors. Inspired by the fact that the Laplacian is reducible in $\mathbb{C}$,

$$\nabla^2 = \partial_{xx} + \partial_{yy} = (\partial_x - \partial_y i)(\partial_x + \partial_y i).$$

we propose an answer in the way of a undetermined coefficients procedure. The problem of factorization reduces to finding a square root and solving a system of linear equations plus a test equation (a first order linear partial differential equation). If the square root exists, the linear system has a unique solution. If in addition the test equation is satisfied, we otain a factorization into linear factors.

In this approach we only need to:

- compute a square root;
- solve a system of two linear equations in two unknowns; and
- evaluate the linear system's solution in a linear partial differential equation.

Our approach provides an algorithmic solution to the problem of factoring second order linear partial differential operators. It is based on naive algebraic methods, and it is general in the sense that it produces the same known results for second order linear ordinary differential operators. In contrast to the ordinary case, for the partial case we need not consider any Riccati equation.

Comparing our approach with other known ones, we have found that we not only propose a possible right factor but rather we find the factorization when it exists, without appealing to the necessity to define new structures or to extend the original domain in which we are working.

We have tried to generalize our approach for higher order operators, however we have found that the situation is rather different. In order to get a possible factorization in a product of lower order operators one needs to solve a system of algebraic and differential equations. But the main difficulty is in fact that the number of algebraic equations is always less than the number of differential equations. In some cases we can plug in the solutions of an overdeterminated algebraic system into the differential equations, however we still have to solve some differential equations.

The chapter is organized in the following way:

- In Section 2.1 we present the basic definitions and the necessary algebraic machinery.
- Section 2.2 presents our approach, which is our main result, a theorem that establishes the bases for factoring second order linear partial differential operators in $k(x, y)[\partial_x, \partial_y]$.
- In Section 2.3 we extend our approach to the case of third order linear differential operators.
- In Section 2.4 we present the recent algorithm "Hensel Decent" for factoring lineal partial differential operators, which reduces to polynomial factorization (over k), rational operations in k and taking derivatives.

2.1 Definitions

In this section we present the algebraic machinery in which the factorization algorithms of linear differential operators can be presented and proved to be correct. The main idea is to define the notion of derivation in a pure algebraic setting (i.e., without using the notions of "function", "limit", and "tangent line" from analysis). This way, we can later translate a factorization problem of linear differential operators to a factorization problem of polynomials in some algebraic structure, which can be done using algebraic algorithms.

Let $\mathcal{R}$ be a commutative ring (resp. field). A **derivation** on $\mathcal{R}$ is a map $\delta : \mathcal{R} \to \mathcal{R}$ such that

$$\delta(a+b) = \delta(a) + \delta(b), \text{ and } \delta(ab) = \delta(a)b + a\delta(b).$$

for all $a, b \in \mathcal{R}$. The pair $(\mathcal{R}, \delta)$ is called a **differential ring (resp. field)**. The set

$$\mathrm{Const}_\delta(\mathcal{R}) = \{c \in \mathcal{R} \text{ such that } \delta(c) = 0\}$$

is called the **subring (resp. subfield) of constants** of $\mathcal{R}$ w.r.t. δ.

A subset $\mathcal{S} \subseteq \mathcal{R}$ is called a **differential subring (resp. subfield)** of $\mathcal{R}$ if $\mathcal{S}$ is a subring (resp. subfield) of $\mathcal{R}$ and $\delta(\mathcal{S}) \subseteq \mathcal{S}$.

The following are examples of differential rings:

- Any ring $\mathcal{R}$ with trivial derivation, i.e., $\delta = 0$, is a differential ring.
- The ring of real C^∞-functions on an open subset $U \subseteq \mathbb{R}$ with ordinary derivation $\frac{d}{dx}$.
- The ring of real C^∞-functions on an open subset $U \subseteq \mathbb{R}^n$ with partial derivation $\frac{\partial}{\partial x_i}$ ($1 \le i \le n$).
- the ring of analytic functions on an open set of $\mathbb{C}$ with complex differentiation.

For a commutative ring $\mathcal{R}$ we also have

- The polynomial ring $\mathcal{R}[t]$ with formal derivation $\frac{d}{dt}$.
- $\mathcal{R}[t_1, \ldots, t_n]$ with formal partial derivation $\frac{\partial}{\partial t_i}$.

The following are examples of differential fields. Let $\mathcal{C}$ denote a field.

- $\mathcal{C}(z)$, with derivation $f \mapsto f' = \frac{df}{dz}$.
- The field of formal Laurent series $\mathcal{C}((z))$, with derivation $f \mapsto f' = \frac{df}{dz}$.
- The field of convergent Laurent series $\mathbb{C}(\{z\})$, with derivation $f \mapsto f' = \frac{df}{dz}$.
- The field of all meromorphic functions on any open connected subset of the extended complex plane $\mathbb{C} \cup \{\infty\}$, with derivation $f \mapsto f' = \frac{df}{dz}$.
- $\mathbb{C}(z, e^z)$, with derivation $f \mapsto f' = \frac{df}{dz}$.

Let $\mathcal{R}$ be a differential ring with derivation $a \mapsto a'$. The **ring of differential polynomials** in $y_1, \ldots, y_n$ over $\mathcal{R}$, denoted by

$$\mathcal{R}\{\{y_1, \ldots, y_n\}\},$$

is defined in the following way. For each $i = 1, \ldots, n$ let

$$y_i^{(j)}, \ j \in \mathbb{N}$$

be an infinite set of distinct indeterminates. For convenience we will write y_i for $y_i^{(0)}$, y_i' for $y_i^{(1)}$ and y_i'' for $y_i^{(2)}$. We define

$$\mathcal{R}\{\{y_1, \dots, y_n\}\},$$

to be the polynomial ring

$$\mathcal{R}[y_1, y_1', y_1'', \dots, y_2, y_2', y_2'', \dots, y_n, y_n', y_n'', \dots].$$

We extend the derivation of $\mathcal{R}$ to a derivation on

$$\mathcal{R}\{\{y_1, \dots, y_n\}\},$$

by setting $\left(y_i^{(j)}\right)' = y_i^{(j+1)}$.

A **Δ-ring** is a commutative ring $\mathcal{R}$ with identity $1_{\mathcal{R}}$ equipped with a set of derivations

$$\Delta = \{\delta_1, \dots, \delta_r\},$$

such that

$$\delta_i \delta_j = \delta_j \delta_i, \text{ for all } i, j = 1, \dots, r.$$

A **Δ-field** k is a field that is a Δ-ring. If $\mathcal{R}$ is a Δ-ring, the set

$$\{c \in \mathcal{R} \mid \delta_i(c) = 0 \text{ for all } i = 1, \dots, r\}$$

is called the **constants** of $\mathcal{R}$. This can be seen to be a ring and, if $\mathcal{R}$ is a field, then this set will be a field as well.

The following are examples of Δ-fields.

- Let $\mathcal{C}$ be a field and $t_1, \dots, t_r$ indeterminates. The field $\mathcal{C}(t_1, \dots, t_r)$ with derivations δ_i, $i = 1, \dots, r$ defined by

$$\delta_i(c) = 0, \text{ for all } c \in \mathcal{C} \text{ and}$$

$$\delta_i(t_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

  is a Δ-field.

- The field of fractions $\mathcal{C}((t_1, \dots, t_r))$ of the ring of formal power series in r variables is a Δ-field with the derivations defined above.

- For $\mathcal{C} = \mathbb{C}$, the complex numbers, the field of fractions

$$\mathbb{C}(\{t_1, \dots, t_r\})$$

  of the ring of convergent power series in r variables with Δ defined as above is again a Δ-field.

Let $\mathcal{R}$ and $\mathcal{S}$ be Δ-rings, and let $\phi : \mathcal{R} \to \mathcal{S}$ be a ring homomorphism. If ϕ commutes with each $\delta \in \Delta$, then ϕ is called a **differential homomorphism (or Δ-homomorphism)**.

Let k be a Δ-field with derivations $\Delta = \{\partial_1, \dots, \partial_r\}$. The **ring of linear partial differential operators** $k[\partial_1, \dots, \partial_r]$ with coefficients in k is the non-commutative polynomial ring in the variables ∂_i, where the ∂_i satisfy

$$\partial_i a = a\partial_i + \partial_i(a) \text{ for all } a \in k.$$

where $\partial_i(a) \in k$ is the derivative of a with respect to ∂_i.

The following are examples of rings of differential operators.

- Let $(k, ')$ be a differential field such that its subfield of constants $\mathcal{C}$ is different from k and has characteristic 0. When $r = 1$ we obtain the skew ring (i.e., non-commutative ring) $\mathcal{D} := k[\partial]$, called the **ring of linear ordinary differential operators** with coefficients in k, which consists of all expressions
$$L := a_n\partial^n + \cdots + a_1\partial + a_0$$
$$\text{with } n \in \mathbb{Z},\ n \geq 0, \text{ and } a_i \in k \text{ for } i = 0, \ldots, n.$$
The degree of L above, denoted by deg L, is m if $a_m \neq 0$ and $a_i = 0$ if $i > m$. In the case $L = 0$ we define the degree to be ∞. The addition in $\mathcal{D}$ is obvious and the multiplication is completely determined by the prescribed rule:
$$\partial a = a\partial + a'.$$
Since there exists an element $a \in k$ with $a' \neq 0$, the ring $\mathcal{D}$ is not commutative. A differential operator
$$L = a_n\partial^n + \cdots + a_1\partial + a_0$$
acts on k with the interpretation
$$\partial(y) := y'.$$
Thus the equation $L(y) = 0$ has the same meaning as the scalar differential equation
$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0.$$
In connection with this one sometimes uses the expression "order of L" and writes $\mathrm{ord}(L)$, instead of the degree of L.

- Let k be a field. The ring $k(x, y)[\partial_x, \partial_y]$ of linear differential operators in the variables ∂_x, ∂_y with coefficients in $k(x, y)$, the field of the rational functions in the indeterminates x, y over k.

2.2 Naive Approach

Let k be an algebraically closed field of characteristic zero. $k(x, y)[\partial_x, \partial_y]$, the ring of linear differential operators in the variables ∂_x, ∂_y.

Theorem 2. *Let*
$$L_2 := \partial_x^2 + E\partial_x\partial_y + D\partial_y^2 + C\partial_x + B\partial_y + A. \tag{2.2.1}$$
be the second order linear partial differential operator with coefficients in $k(x, y)$. The operator L_2 splits into linear factors
$$[\partial_x + \beta\partial_y + \alpha]\,[\partial_x + \psi\partial_y + \omega] \tag{2.2.2}$$
if and only if
$$\beta = \frac{E}{2} - \frac{\sqrt{E^2 - 4D}}{2},\ \psi = \frac{E}{2} + \frac{\sqrt{E^2 - 4D}}{2},$$
and the following system is solvable
$$\begin{cases} \alpha + \omega &= C \\ \partial_x(\psi) + \beta\partial_y(\psi) + \beta\omega + \alpha\psi &= B\ . \\ \partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega &= A. \end{cases} \tag{2.2.3}$$

Proof. Suppose that there exist $\alpha, \beta, \psi, \omega \in k(x, y)$ such that

$$L_2 = [\partial_x + \beta\partial_y + \alpha]\,[\partial_x + \psi\partial_y + \omega]\,.$$

Now, developing the right hand side

$$\partial_x^2 + (\beta + \psi)\partial_x\partial_y + \beta\psi\partial_y^2 + (\alpha + \omega)\partial_x +$$

$$[\partial_x(\psi) + \beta\partial_y(\psi) + \beta\omega + \alpha\psi]\,\partial_y +$$

$$\partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega$$

Comparing the second and third coefficients with original operator we get

$$\beta + \psi = E \text{ and } \beta\psi = D.$$

Hence,

$$\beta^2 - E\beta + D = 0$$

Therefore,

$$\beta = \frac{E}{2} - \frac{\sqrt{E^2 - 4D}}{2},\ \psi = \frac{E}{2} + \frac{\sqrt{E^2 - 4D}}{2},$$

Comparing the rest of the coefficients with the original operator we obtain the following system of equations

$$\begin{cases} \alpha + \omega & = & C \\ \partial_x(\psi) + \beta\partial_y(\psi) + \beta\omega + \alpha\psi & = & B \\ \partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega & = & A \end{cases}.$$

If we can solve this system of the first two equations for α and ω (system (2.2.3)), then we can have a factorization of the operator (2.2.1) in the form (2.2.2), otherwise the original operator has not this kind of factorization in $k(x, y)$.

□

Now from the previous theorem we can extract a procedure that we called Pseudo-algorithm $gl1$.

Algorithm $gl1$

Input: A second order linear partial differential operator

$$L_2 := \partial_x^2 + E\partial_x\partial_y + D\partial_y^2 + C\partial_x + B\partial_y + A \tag{2.2.4}$$

with $A, B, C, D, E \in k(x, y)$.

Output: A factorization of the form:

$$\left[\partial_x + \left(\frac{E}{2} - Z\right)\partial_y + \alpha\right]\left[\partial_x + \left(\frac{E}{2} + Z\right)\partial_y + \omega\right]. \tag{2.2.5}$$

1. Compute

$$Z = \frac{\sqrt{E^2 - 4D}}{2}. \tag{2.2.6}$$

2. If $\sqrt{E^2 - 4D}$ does not exist in $k(x, y)$ go to step 10.

3. Solve the system

$$\begin{cases} \alpha + \omega & = & C \\ \partial_x\left(\frac{E}{2} + Z\right) + \left(\frac{E}{2} - Z\right)\partial_y\left(\frac{E}{2} + Z\right) + & & \\ \left(\frac{E}{2} - Z\right)\omega + \left(\frac{E}{2} + Z\right)\alpha & = & B \end{cases} \tag{2.2.7}$$

where α and ω are the unknowns.

4. If System (2.2.7) is not solvable go to step 10.
5. If System (2.2.7) has unique solution, then test the solution in the equation:

$$\partial_x(\omega) + \left(\frac{E}{2} - Z\right)\partial_y(\omega) + \alpha\omega = A. \tag{2.2.8}$$

6. If Equation (2.2.8) is not satisfied go to step 10.
7. Substitute E, Z, α and ω in Expression (2.2.5).
8. Return Expression (2.2.5) and go to step 11.
9. If System (2.2.7) has infinitely many solutions then return "There is no decision", and go to step 11.
10. Return "The operator (2.2.4) has not non-trivial factorization of the form (2.2.5) in $k(x,y)[\partial_x, \partial_y]$".
11. End.

Remark 2.2.1. *Note that if we are lucky we can compute the step* 1 *in* $k(x,y)$, *however we can always compute the step* 3 *in* k, *and the step* 5 *is completely finite because in* $k(x,y)$ *we can decide the equality. If System (2.2.7) has infinitely many solutions then we can not decide if there exists a factorization of the form (2.2.5), because in that case we should solve the differential equation (2.2.8). By the proof of Theorem (2) the algorithm is correct, and for the structure of itself it is completely clear that the algorithm terminates.*

Now we will show how the theorem (2) works in the application of algorithm $gl1$ to the next example taken from Grigoriev and Schwarz [2004].

Example 1. *Let* $L \in \mathbb{C}(x,y)[\partial_x, \partial_y]$ *be*

$$\partial_x^2 + \frac{1}{x}(x^2y^2 + x - y)\partial_x\partial_y + \frac{y}{x}(x^2y - 1)\partial_y^2 - x(y-1)\partial_x +$$
$$\frac{1}{x^2}(x^4y^2 + x^3y + x^2y^2 - x^2y - x + y)\partial_y - x^2y - x - y$$

where

$$E = \frac{1}{x}(x^2y^2 + x - y),\ D = \frac{y}{x}(x^2y - 1),\ C = -x(y-1),$$
$$B = \frac{1}{x^2}(x^4y^2 + x^3y + x^2y^2 - x^2y - x + y),\ \textit{and}\ A = -x^2y - x - y.$$

By Equation (2.2.6) we have

$$Z = \frac{1}{2x}(x^2y^2 - x - y).$$

By Equation (2.2.5) we are searching for a factorization of the form

$$(\partial_x + \partial_y + \alpha)\left[\partial_x + \left(xy^2 - \frac{y}{x}\right)\partial_y + \omega\right].$$

By System (2.2.7) we get

$$\begin{cases} \alpha + \omega & = & -x(y-1) \\ y^2 + \dfrac{y}{x^2} + 2xy - \dfrac{1}{x} + \omega + xy^2\alpha - \dfrac{y}{x}\alpha & = & \dfrac{1}{x^2}(x^4y^2 + x^3y + x^2y^2 \Rightarrow \\ & & -x^2y - x + y) \end{cases}$$

$$\begin{cases} \alpha + \omega & = & x - xy \\ \omega + xy^2\alpha - \dfrac{y}{x}\alpha & = & x^2y^2 - xy - y \end{cases}.$$

The solution of this system is:

$$\alpha = x \text{ and } \omega = -xy.$$

Now, testing this solution in Equation (2.2.8) we obtain in fact that

$$\partial_x(\omega) + \left(\frac{E}{2} - Z\right)\partial_y(\omega) + \alpha\omega = -y - x - x^2y.$$

Therefore, the factorization of L in $\mathbb{C}(x,y)[\partial_x, \partial_y]$ *is:*

$$(\partial_x + \partial_y + x) \cdot \left[\partial_x + \left(xy^2 - \frac{y}{x}\right) \cdot \partial_y - xy\right].$$

Remark 2.2.2. *As we have seen above, the last procedure provides a computational algebraic approach to the problem of fatorization of linear partial differential operators. But despite of it with the same ideas we can not factorize second order ordinary homogeneous operators in the same way, because if we consider a second order ordinary operator*

$$L_o = \partial^2 + C\partial + A$$

and we want a factorization of the form

$$(\partial + \alpha)(\partial + \omega)$$

applying the same procedure we get the system

$$\begin{cases} \alpha + \omega & = & C \\ \partial(\omega) + \alpha\omega & = & A \end{cases}. \tag{2.2.9}$$

If we substitute ω *from the first equation of this system into the second one we obtain the following Riccati equation*

$$\alpha' + \alpha^2 - C\alpha - C' + A = 0.$$

Althout with this approach we cannot solve Riccati equations we obtain the same reduction procedure from second order differential equation to a Riccati equation, in Chapter 2 we dedicate an entire section to the computational algebraic methods for solving certain particular Riccati equations, namely Section 3.3.

2.3 Extension of the Order: Case Third Order

In this section we generalize our approach to operators of higher order, in particular to third order linear differential operators, and we will see that the problem of factoring higher order operators reduces to solveing a system of algebraic and differential equations.

As the ring of linear partial differential operators is not commutative, for third order operators we have two possible factorizations in a product of lower order operators, as we will see in the following theorem that we present without proof.

Finally, we would like to mention that if we act recursively applying the algorithm $gl1$, then we can find a factorization of a third order operator into linear factors, if it exists.

Theorem 3. *Let*

$$L_3 := \partial_x^3 + J\partial_x^2\partial_y + H\partial_x\partial_y^2 + G\partial_y^3 + F\partial_x^2 + E\partial_x\partial_y + D\partial_y^2 + C\partial_x + B\partial_y + A \tag{2.3.10}$$

be a linear differential operator of third order with coefficients in $k(x, y)$.

1. *The operator* L_3 *can be factorized in the following way*

$$\left(\partial_x^2 + \epsilon\partial_x\partial_y + \delta\partial_y^2 + \gamma\partial_x + \beta\partial_y + \alpha\right)(\partial_x + \psi\partial_y + \omega) \tag{2.3.11}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \psi$ *and* ω *are undetermined variables in* $\mathbb{C}(x, y)$, *if and only if the following system is solvable*

$$\begin{cases} \psi + \epsilon &= J \\ \epsilon\psi + \delta &= H \\ \delta\psi &= G \\ \omega + \gamma &= F \\ 2\partial_x(\psi) + \epsilon\partial_y(\psi) + \epsilon\omega + \gamma\psi + \beta &= E \\ \epsilon\partial_x(\psi) + 2\delta\partial_y(\psi) + \delta\omega + \delta + \beta\psi &= D \\ 2\partial_x(\omega) + \epsilon\partial_y(\omega) + \gamma\omega + \alpha &= C \\ \partial_{xx}(\psi) + \epsilon\partial_{xy}(\psi) + \delta\partial_{yy}(\psi) + \gamma\partial_x(\psi) + & \\ \epsilon\partial_x(\omega) + \beta\partial_y(\psi) + 2\delta\partial_y(\omega) + \beta\omega + \alpha\psi &= B \\ \partial_{xx}(\omega) + \epsilon\partial_{xy}(\omega) + \gamma\partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega &= A \end{cases} . \tag{2.3.12}$$

2. *The operator* L_3 *can be factorized of the following way*

$$(\partial_x + \beta\partial_y + \alpha)\left(\partial_x^2 + \sigma\partial_x\partial_y + \tau\partial_y^2 + \phi\partial_x + \psi\partial_y + \omega\right) \tag{2.3.13}$$

where $\alpha, \beta, \sigma, \tau, \phi, \psi$ *and* ω *are indeterminates variable en* $k(x, y)$, *if and only if the following system is solvable*

$$\begin{cases} \sigma + \beta &= J \\ \tau + \beta\sigma &= H \\ \beta\tau &= G \\ \phi + \alpha &= F \\ \partial_x(\sigma) + \psi + \beta\partial_y(\sigma) + \beta\phi + \alpha\sigma &= E \\ \partial_x(\tau) + \beta\partial_y(\tau) + \beta\psi + \alpha\tau &= D \\ \partial_x(\phi) + \omega + \beta\partial_y(\phi) + \alpha\phi &= C \\ \partial_x(\psi) + \beta\partial_y(\psi) + \beta\omega + \alpha\psi &= B \\ \partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega &= A \end{cases} . \tag{2.3.14}$$

Remark 2.3.1. *In this case, due to the derivation rules of the definition ring of operators, it is much more difficult to obtain a closed form of the possible factorization, moreover if one wants to extract a term from the left. However, in this case we could propose a factorization of the form:*

$$[\partial_x^2 + (J+H)\partial_x\partial_y + G\partial_y^2 + F\partial_y + (E+D)\partial_y + \alpha](\partial_x + \partial_y + \omega).$$

It is clear that this is a particular factorization, however it gave us the idea that it could be also possible to factor in the same way as the second order case.

Example 2. *Blumberg [1912]*

Let

$$\partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + (x+2)\partial_y$$

be the third order operator with

$$J = x,\ H = 0,\ G = 0,\ F = 2,\ E = 2(x+1),\ D = 0,$$

$$C = 1,\ B = x+2,\ \textit{and}\ A = 0.$$

By Equation (2.3.11) we are searching a factorization of the form

$$\left(\partial_x^2 + \epsilon\partial_x\partial_y + \delta\partial_y^2 + \gamma\partial_x + \beta\partial_y + \alpha\right)\left(\partial_x + \psi\partial_y + \omega\right).$$

By System (2.3.12) we have

$$\begin{cases} \psi + \epsilon & = & x \\ \epsilon\psi + \delta & = & 0 \\ \delta\psi & = & 0 \\ \gamma + \omega & = & 2 \\ 2\partial_x(\psi) + \epsilon\partial_y(\psi) + \epsilon\omega + \gamma\psi + \beta & = & 2(x+1) \\ \epsilon\partial_x(\psi) + 2\delta\partial_y(\psi) + \delta\omega + \delta + \beta\psi & = & 0 \\ 2\partial_x(\omega) + \epsilon\partial_y(\omega) + \gamma\omega + \alpha & = & 1 \\ \partial_{xx}(\psi) + \epsilon\partial_{xy}(\psi) + \delta\partial_{yy}(\psi) + \gamma\partial_x(\psi) + & & \\ \epsilon\partial_x(\omega) + \beta\partial_y(\psi) + 2\delta\partial_y(\omega) + \beta\omega + \alpha\psi & = & x+2 \\ \partial_{xx}(\omega) + \epsilon\partial_{xy}(\omega) + \gamma\partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega & = & 0 \end{cases}.$$

By the fourth the equation, if $\psi = 0$ then

$$\begin{cases} \epsilon & = & x \\ \delta & = & 0 \\ \gamma + \omega & = & 2 \\ \epsilon\omega + \beta & = & 2(x+1) \Rightarrow \\ 2\partial_x(\omega) + \epsilon\partial_y(\omega) + \gamma\omega + \alpha & = & 1 \\ \epsilon\partial_x(\omega) + 2\delta\partial_y(\omega) + \beta\omega & = & x+2 \\ \partial_{xx}(\omega) + \epsilon\partial_{xy}(\omega) + \gamma\partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega & = & 0 \end{cases}$$

$$\begin{cases} \gamma & = & 2-\omega \\ x\omega+\beta & = & 2(x+1) \\ 2\partial_x(\omega)+x\partial_y(\omega)+\gamma\omega+\alpha & = & 1 \\ x\partial_x(\omega)+\beta\omega & = & x+2 \\ \partial_{xx}(\omega)+x\partial_{xy}(\omega)+\gamma\partial_x(\omega)+\beta\partial_y(\omega)+\alpha\omega & = & 0 \end{cases} \Rightarrow$$

$$\begin{cases} \gamma & = & 2-\omega \\ x\omega+\beta & = & 2(x+1) \\ 2\partial_x(\omega)+x\partial_y(\omega)+2\omega-\omega^2+\alpha & = & 1 \\ x\partial_x(\omega)+\beta\omega & = & x+2 \\ \partial_{xx}(\omega)+x\partial_{xy}(\omega)+(2-\omega)\partial_x(\omega)+\beta\partial_y(\omega)+\alpha\omega & = & 0 \end{cases}$$

which has solution

$$\omega = 1 \ \Rightarrow\ \gamma = 1,\ \beta = x+2 \text{ and } \alpha = 0.$$

Where,

$$\left[\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+2)\partial_y\right](\partial_x + 1).$$

Now, working in a recursive way, let us consider the left factor

$$\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y$$

with

$$\bar{E} = x,\ \bar{D} = 0,\ \bar{C} = 1,\ \bar{B} = x+2, \text{ and } \bar{A} = 0.$$

By Equation (2.2.6) we have

$$\bar{Z} = \frac{x}{2}.$$

By Expression (2.2.5) we are searching, for this second order operator, a factorization of the form

$$(\partial_x + \bar{\alpha})\left[\partial_x + x\partial_y + \bar{\omega}\right].$$

By System (2.2.7) we obtain

$$\begin{cases} \bar{\alpha}+\bar{\omega} & = & 1 \\ 1+x\bar{\alpha} & = & x+2 \end{cases}$$

which has solution:

$$\bar{\alpha} = 1 + \frac{1}{x} \text{ and } \bar{\omega} = -\frac{1}{x}.$$

Substituting the solution in Equation (2.2.8) we get

$$\partial_x(\bar{\omega}) + \left(\frac{\bar{E}}{2} - \bar{Z}\right)\partial_y(\bar{\omega}) + \bar{\alpha}\bar{\omega} \neq 0.$$

Hence, the solution of the system (2.2.7) does not satisfy Equation (2.2.8). By Theorem (2), the left factor does not have factorization in linear factors in $\mathbb{C}(x,y)[\partial_x,\partial_y]$. Therefore,

$$\partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + (x+2)\partial_y =$$

$$\left[\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+2)\partial_y\right](\partial_x + 1).$$

Example 3. *Let us consider again the third order operator of Example (2)*

$$\partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + (x+2)\partial_y$$

we know that

$$J = x,\ H = 0,\ G = 0,\ F = 2,\ E = 2(x+1),\ D = 0,$$
$$C = 1,\ B = x+2,\ \text{and } A = 0.$$

By Equation (2.3.13) we are searching first for a factorization of the form

$$(\partial_x + \beta\partial_y + \alpha)\left(\partial_x^2 + \sigma\partial_x\partial_y + \tau\partial_y^2 + \phi\partial_x + \psi\partial_y + \omega\right).$$

By System (2.3.14) we get

$$\begin{cases} \sigma + \beta & = & x \\ \tau + \beta\sigma & = & 0 \\ \beta\tau & = & 0 \\ \phi + \alpha & = & 2 \\ \partial_x(\sigma) + \psi + \beta\partial_y(\sigma) + \beta\phi + \alpha\sigma & = & 2(x+1) \\ \partial_x(\tau) + \beta\partial_y(\tau) + \beta\psi + \alpha\tau & = & 0 \\ \partial_x(\phi) + \omega + \beta\partial_y(\phi) + \alpha\phi & = & 1 \\ \partial_x(\psi) + \beta\partial_y(\psi) + \beta\omega + \alpha\psi & = & x+2 \\ \partial_x(\omega) + \beta\partial_y(\omega) + \alpha\omega & = & 0 \end{cases}.$$

By the fourth equation, if $\beta = 0$ then

$$\begin{cases} \sigma & = & x \\ \tau & = & 0 \\ \phi + \alpha & = & 2 \\ \partial_x(\sigma) + \psi + \alpha\sigma & = & 2(x+1) \\ \partial_x(\phi) + \omega + \alpha\phi & = & 1 \\ \partial_x(\psi) + \alpha\psi & = & x+2 \\ \partial_x(\omega) + \alpha\omega & = & 0 \end{cases}.$$

Where,

$$\begin{cases} \phi + \alpha & = & 2 \\ 1 + \psi + x\alpha & = & 2(x+1) \\ \partial_x(\phi) + \omega + \alpha\phi & = & 1 \\ \partial_x(\psi) + \alpha\psi & = & x+2 \\ \partial_x(\omega) + \alpha\omega & = & 0 \end{cases}$$

which has solution

$$\alpha = 1 \ \Rightarrow\ \phi = 1,\ \psi = x+1 \text{ and } \omega = 0.$$

Hence, we can factor the operator in the form

$$\partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + (x+2)\partial_y =$$
$$(\partial_x + 1)\left[\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+1)\partial_y\right].$$

Now, working in a recursive way, let us consider the right factor

$$\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+1)\partial_y$$

with

$$\bar{E} = x,\ \bar{D} = 0,\ \bar{C} = 1,\ \bar{B} = x+1,\ \textit{and}\ \bar{A} = 0.$$

By Equation (2.2.6) we have

$$\bar{Z} = \frac{x}{2}.$$

By Expression (2.2.5) we are searching, for the second order operator, a factorization of the form

$$(\partial_x + \bar{\alpha})\left[\partial_x + x\partial_y + \bar{\omega}\right].$$

By System (2.2.7) obtain

$$\begin{cases} \bar{\alpha} + \bar{\omega} & = & 1 \\ 1 + x\bar{\alpha} & = & x+1 \end{cases}$$

which has solution:

$$\bar{\alpha} = 1\ \textit{and}\ \bar{\omega} = 0.$$

Substituting the solution in the Equation (2.2.8) we get in fact that

$$\partial_x(\bar{\omega}) + \left(\frac{\bar{E}}{2} - \bar{Z}\right)\partial_y(\bar{\omega}) + \bar{\alpha}\bar{\omega} = 0.$$

Therefore, the factorization of the right factor in linear factors in $\mathbb{C}(x,y)[\partial_x, \partial_y]$ *is:*

$$\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+1)\partial_y = (\partial_x + 1)(\partial_x + x\partial_y).$$

In conclusion, the factorization of the original operator in linear factors in $\mathbb{C}(x,y)[\partial_x, \partial_y]$ *is:*

$$\partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + (x+2)\partial_y = (\partial_x + 1)(\partial_x + 1)(\partial_x + x\partial_y).$$

2.4 The Hensel Descent Algorithm

In this section the recent algorithm due to Grigoriev and Schwarz [2004] is discussed. It is named Hensel descent because it is close in nature to the well-known Hensel lifting used widely in polynomial factorization. The main difference is that in case of differential operators one has to compute the coefficients starting with the highest derivatives going to the lowest because in the product of operators the coefficients of higher derivatives in the factors contribute to the coefficients of lower derivatives in the product.

Grigoriev and Schwarz define the **symbol of an operator** as the homogeneous polynomial with the same coefficients as the homogeneous part and the same powers than the corresponding derivatives. They call an operator **separable** if its symbol is separable, i.e., if all its roots in a splitting field are distinct.

If an operator is separable then to finding a factorization reduces to polynomial factorization in the field of coefficients.

Let k be a Δ-field and let $\mathcal{D} = k[\partial_1, \ldots, \partial_m]$ be the non-commutative polynomial ring of linear (partial) differential operators in the variables ∂_i with coefficients in k.

For a derivative

$$\partial^J = \partial_1^{j_1} \dots \partial_r^{j_1}$$

denote its order

$$ord(\partial^J) = |J| = j_1 + \cdots + j_m.$$

For a linear partial differential operator $L \in \mathcal{D}$ denote

$$L = \sum_{|J|=r} a_J \partial^J + \sum_{|J|<r} b_J \partial^J$$

of the order $\text{ord}(L) = r$.

Let us consider the polynomial ring $k[Z_1, \dots, Z_m]$ in the algebraic indeterminates Z_k. The **symbol** of L, denoted by $s(L)$ is homogeneous polynomial defined by

$$s(L) = \sum_{|J|=r} a_j Z^J \in k[Z_1, \dots, Z_m].$$

For example, if $L_2 \in \mathbb{Q}(x,y)[\partial_x, \partial_y]$ given by

$$L_2 = \frac{2y}{x^2}\partial_x^2 + 2(x+1)\partial_x\partial_y + (x+2)\partial_y^2 + 2\partial_x - \frac{2y}{x}\partial_y + \frac{y^2}{x^2}(1 - x^4y^2),$$

then $s(L_2) \in \mathbb{Q}(x,y)[Z_1, Z_2]$, is the polynomial

$$s(L_2) = \frac{2y}{x^2}Z_1^2 + 2(x+1)Z_1Z_2 + (x+2)Z_2^2.$$

The operator L is called **separable** if $s(L)$ is separable.

The Hensel Descent Algorithm

Input: A separable operator $L \in \mathcal{D}$ of order r given by

$$L = \sum_{|J|=r} a_J \partial^J + \sum_{|J|<r} b_J \partial^J.$$

Then,

$$s(L) = gh$$

where

$$g = \sum_J g_J Z^J, h = \sum_J h_J Z^J \in F[Z_1, \dots, Z_m]$$

are homogeneous polynomials of degrees

$$\deg(g) = k, \ \deg(g) = k, \ \text{with } k + l = r.$$

Output: A factorization in the form:

$$L = \left(\sum_J g_J \partial^J + \sum_{0 \le j \le k-1} G_j \right) \left(\sum_J h_J \partial^J + \sum_{0 \le j \le l-1} H_j \right) \qquad (2.4.15)$$

where

$$G_j = \sum_{|J|=j} g_{J,j} \partial^J, \ \text{and } H_j = \sum_{|J|=j} h_{J,j} \partial^J$$

contain only the derivatives or the order j.

Denote the corresponding homogeneous polynomials of degrees j by

$$g_j = \sum_{|J|=j} g_{J,j} Z^J, \ \text{and } h_j = \sum_{|J|=j} h_{J,j} Z^J.$$

Proceed recursively decreasing the order in the following way:

- Suppose that G_{j_1}, H_{j_2} are already constructed with

$$j_1 \geq \max\{t-l+1,0\}, \text{ and } j_2 \geq \max\{t-k+1,0\}$$

for certain $0 \leq t \leq r-1$ (at the first step of the recursion set $t = r-1$).

- Compare the coefficients of the derivatives of order t in both sides of (2.4.15).

- Rewrite the the right-hand side in terms of the corresponding homogeneous polynomials of degree t and obtain

$$gh_{t-k} + hg_{t-l} + p$$

(provided that $t \geq k$, $t \geq l$) where the coefficients of the homogeneous polynomial p are already known being the rational expressions of the derivatives of the coefficients of the already constructed G_{j_1}, H_{j_2}.

Since $t-k < l$, due to the separability of L the polynomials g, h are relatively prime, we can conclude that there exists at most one pair of polynomials g_{t-l}, h_{t-k} which yields a known polynomial

$$q = gh_{t-k} + hg_{t-l}.$$

- Look for g_{t-l}, h_{t-k} by means solving a linear algebraic system in the coefficients of g_{t-l}, h_{t-k}.

- If the system is unfeasible then halt and say that the polynomial factorization $s(L) = gh$ does not lead to a factorization of L.

- Otherwise, output g_{t-l}, h_{t-k} and continue the recursion.

In the case when $t < k$ (or $t < l$, respectively) the polynomial h_{t-k} is absent (or g_{t-l} is absent, respectively).

- In the case when both $t < k$, $t < l$ verify whether the coefficients at the derivatives of order t in both sides of (2.4.15) coincide. And again halt if this fails.

Part II

FACTORIZATION OF LINEAR ORDINARY DIFFERENTIAL OPERATORS

3. THE BEKE ALGORITHM

In 1894 Beke gave a method for factoring linear differential operators in the ring $\overline{\mathbb{Q}}(x)[\partial]$, and after almost one hundred years it has been improved and extended to the ring $k(x)[\partial]$ where k is an arbitrary differential field of characteristic 0. It has also been implemented in Computer Algebra Systems by Schwarz [1989], Schwarz [1994], Bronstein [1994], Bronstein and Petkovšek [1996].

Schwarz analyses the costs of factoring linear homogeneous differential equations with rational coefficients and he describes the algorithm of Beke in a different way by recursively reducing the order of possible right factors. Moreover, he estimates bounds for the degree of their coefficients and he computes the size of rational solutions of certain differential equations. Finally, he describes how the algorithm LODEF is implemented in the computer algebra system Scratchpat II.

The first section of this chapter is devoted to some basic preliminaries about differential Galois theory of linear homogeneous differential equations. Subsequently, Sections 3.2 and 3.3 are developed to study the methods for finding rational and exponential solutions of linear homogeneous differential equations. In section 3.4 we will present the Beke's algorithm. In section 3.5 and 3.6 we present some variants of the Beke's algorithm, namely the Schwarz's LODEF algorithm and the Beke-Bronstein algorithm, respectively.

3.1 Preliminaries

In ordinary Galois theory of algebraic equations, questions about solvability of equations are translated into questions about fields and finite groups. For differential equations, the proper setting is differential fields and algebraic groups.

The goal of Differential Galois Theory is a Fundamental Theorem which sets up a bijective correspondence between the intermediate differential subfields of an extension of differential fields and certain subgroups of the group of differential automorphisms of the field extension (the differential Galois group).

Let (k,δ) be a differential field. We also write $y^{(n)}$ instead of $\delta^n(y)$ and $y', y'', \ldots$ for $\delta(y), \delta^2(y), \ldots$. The field of constants

$$\mathrm{Const}_\delta(k) = \{c \in k \mid c' = 0\}$$

is denoted by $\mathcal{C}$. A **differential field extension** of (k,δ) is a differential field (K,Δ) such that K is a field extension of k and Δ is an extension of the derivation of k to the derivation on K. An **order** n **linear scalar differential equation** over k is an equation of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = b \tag{3.1.1}$$

where $a_i, b \in k$. The equation is called **homogeneous** if $b = 0$, and **inhomogeneous otherwise**. A **solution** of (3.1.1) in a differential extension $K \supseteq k$ is an element $f \in K$ such that

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1f' + a_0f = b.$$

A differential field extension (K,Δ) of (k,δ) is called a **Liouvillian extension** if there is a tower of fields

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K$$

where K_{i+1} is a simple field extension $K_i(\eta_i)$ of K_i, such that one of the following holds:

- η_i is algebraic over K_i, or
- $\eta_i' \in K_i$ (extension by an integral), or
- $\eta_i'/\eta_i \in K_i$ (extension by the exponential of an integral).

A solution of $L(y) = 0$ which is contained in

- k, the coefficient field, will be called a **rational solution**,
- an algebraic extension of k will be called an **algebraic solution**,
- a Liouvillian extension of k will be called a **Liouvillian solution**.

A solution z of $L(y) = 0$ is called **exponential** if z'/z is in the coefficient field k.

Let $A \in k^{n\times n}$ be an $n \times n$ matrix with entries in the field k. A **linear system** is a vector equation of the form

$$y' = Ay. \tag{3.1.2}$$

A **solution** of (3.1.2) in a differential extension $K \supseteq k$ is an element $v \in K^n$ such that $v' = Av$.

The solution set of a linear system (3.1.2) in a given extension $K \supseteq k$ is a vector space over $\mathcal{C}$. The same is true for homogeneous linear scalar equations. Practically, both concepts describe the same situation.

The **companion matrix** of a homogeneous scalar linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$$

is the matrix

$$A_L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & a_{n-1} \end{pmatrix}.$$

In the following lemma we will see the relation between scalar equations and linear systems.

Lemma 1. *Let $j : k \longrightarrow k^n$ be the map*

$$f \mapsto (\,f, f', f^{(2)}, \dots, f^{(n-1)}\,)^T.$$

For any scalar equation $L(y) = 0$, the map j induces a $\mathcal{C}$-linear isomorphism

$$\{\,f \in k \mid L(f) = 0\,\} \cong \{\,v \in k^n \mid v' = A_L v\,\}.$$

Proof. Let us write L in the form

$$L = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y$$

If $L(f) = 0$ then

$$j(f)' = A_L \cdot j(f).$$

Conversely, if $v' = A_L v$ then

$$v_2 = v_1', v_3 = v_1^{(2)}, \dots, v_n = v_1^{(n-1)} \text{ and } v_n' = -a_0v_1 - a_1v_2 - \cdots - a_{n-1}v_n,$$

whence $L(v_1) = 0$ and $j(v_1) = v$.

□

Therefore, any homogeneous linear differential equation can be considered as a linear system. The following lemma describes the relation between linear dependency over the ground field k and its subfield of constants $\mathcal{C}$.

Lemma 2. *Let $A \in k^{n\times n}$ and consider solutions $v_1, v_2, \ldots, v_r \in k^n$ of the system $y' = Ay$. If $\{v_1, \ldots, v_r\}$ is linearly dependent over k then also over $\mathcal{C}$.*

Proof. By induction over r. For $r = 1$ the statement is true, so let $r > 1$ and assume that $v_1, \ldots, v_r$ are linearly dependent over k. Then

$$\sum_{i=1}^{r} \lambda_i v_i = 0, \quad \lambda_i \in k, \text{ not all } \lambda_i = 0.$$

If a proper subset of $\{v_1, \ldots, v_r\}$ is linearly dependent over k then by induction hypothesis it is linearly dependent over $\mathcal{C}$. So assume that all proper subsets are linearly independent. This implies that

$$\lambda_i \neq 0, \text{ for all } i = 1, \ldots, r,$$

and so

$$v_1 = \sum_{i=2}^{r} -\frac{\lambda_i}{\lambda_1} v_i.$$

Writing $\alpha_i = -\dfrac{\lambda_i}{\lambda_1}$ we get

$$0 = v_1' - Av_1 = \sum_{i=2}^{r} (\alpha_i' v_i + \alpha_i v_i') - Av_1 = \sum_{i=2}^{r} \alpha_i' v_i + A \sum_{i=2}^{r} \alpha_i v_i - Av_1 = \sum_{i=2}^{r} \alpha_i' v_i$$

and thus $\alpha_2' = \cdots = \alpha_r' = 0$, which means that $\alpha_2, \ldots, \alpha_r \in \mathcal{C}_k$. Therefore,

$$v_1 - \alpha_2 v_2 - \cdots - \alpha_r v_r = 0$$

shows that $v_1, \ldots, v_r$ are linearly dependent over $\mathcal{C}_k$.

□

Corollary 4. *$A \in k^{n\times n}$, $K \supseteq k$ with const$(K) = \mathcal{C}$. Then*

$$\dim_{\mathcal{C}} \{\, x \in K^n \mid x' = Ax \,\} \leq n.$$

Consider a matrix $A \in k^{n\times n}$, and assume for a moment, that the system $y' = Ay$ admits n $\mathcal{C}$-linearly independent solutions $v_1, \ldots, v_n \in k^n$. Then the matrix $F = (v_1, \ldots, v_n)$ is non-singular and $F' = AF$.

Let $K \supseteq k$ be a differential extension with $\text{Const}(K) = \mathcal{C}$, $A \in k^{n\times n}$. A matrix $F \in GL_n(K)$ is called a **fundamental matrix** of the system $y' = Ay$ if $F' = AF$.

The **Wronskian matrix** of $y_1, \ldots, y_n \in k$ is the $n \times n$ matrix:

$$W(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

The **Wronskian**, $w(y_1, \ldots, y_n)$ of $y_1, \ldots, y_n$ is $\det(W(y_1, \ldots, y_n))$.

Theorem 5. *Let k be a differential field with field of constants $\mathcal{C}$. Then n elements of k are linearly dependent over $\mathcal{C}$ if and only if their Wronskian vanishes.*

Proof. Suppose $y_1, \ldots, y_n$ are linearly dependent over $\mathcal{C}$, then there exist $c_i \in \mathcal{C}$ for $i = 1, \ldots, n$ not all zero such that

$$c_1 y_1 + \ldots + c_n y_n = 0.$$

On differentiating this equation $n-1$ times we get the n linear homogeneous equations for $c_1, \ldots, c_n$:

$$\begin{aligned} c_1 y_1' + \ldots + c_n y_n' &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)} + \ldots + c_n y_n^{(n-1)} &= 0. \end{aligned}$$

There are thus n equations to determine the constants $c_1, \ldots, c_n$. Since the c_i are not all 0, the determinant must vanish.

Conversely, suppose that the Wronskian of $y_1, \ldots, y_n$ vanishes. By induction we can construct a monic scalar differential equation $L(y) = 0$ of order n over k such that

$$L(y_i) = 0 \text{ for } i = 1, \ldots, n.$$

For $n = 1$, put

$$L_1(y) = y' - \frac{y_1'}{y_1} y,$$

where the term $\frac{y_1'}{y_1}$ is interpreted as 0 if $y_1 = 0$. Suppose, by induction hypothesis, that $L_m(y)$ for $m \geq 1$, has been constructed such that

$$L_m(y_i) = 0 \text{ for } i = 1, \ldots, m.$$

Define now

$$L_{m+1}(y) = L_m(y)' - \frac{L_m(y_{m+1})'}{L_m(y_{m+1})} L_m(y)$$

where the term $\frac{L_m(y_{m+1})'}{L_m(y_{m+1})}$ is interpreted as 0 if $L_m(y_{m+1}) = 0$. Whence,

$$L_{m+1}(y_i) = 0 \text{ for } i = 1, \ldots, m+1.$$

Then $L = L_n$ has the required property. The columns of the Wronskian matrix are solutions of the associated companion matrix differential equation $y' = A_L y$. By Lemma (2), $y_1, \ldots, y_n$ are linearly dependent over $\mathcal{C}$.

□

A set of n solutions $\{y_1, \ldots, y_n\}$ of an order n equation $L(y) = 0$, linearly independent over the constants $\mathcal{C}$, is a **fundamental set or fundamental system** [1] of solutions of $L(y) = 0$.

Let $k \subset K_1$ and $k \subset K_2$ be extensions of differential fields. A field isomorphism $\sigma : K_1 \to K_2$ is a **differential k-isomorphism** if

$$(\sigma(a))' = \sigma(a'), \text{ for all } a \in K_1 \text{ and}$$

$$\sigma(a) = a \text{ for all } a \in k.$$

The **differential Galois group** of a differential extension K of k, denoted by $G(K/k)$, is the set of all k-automorphisms of K.

A differential extension field K of k is called a **Picard-Vessiot extension** of k for the equation $L(y) = 0$ if:

[1] The term **fundamental system** is due to Fuchs, J. für Math. 66 (1866), p. 126 [Ges. Math. Werke 1, p.165]

1. K is generates over k as a differential field by a fundamental set of solutions $\{y_1, \ldots, y_n\}$ of $L(y) = 0$, i.e., $K = k < y_1, \ldots, y_n >$;
2. K has the same field of constants as k, i.e., $\text{const}(K) = \mathcal{C}$.

In other words, a Picard-Vessiot extension field K is the smallest differential field extension of k such that the equation has solution space dimension n over $\mathcal{C}$. The Picard-Vessiot extension is the equivalent of the splitting field for an algebraic equation. If $\mathcal{C}$ is is algebraically closed and of characteristic 0, then there is always a Picard-Vessiot extension, unique up to differential isomorphisms. See Kaplansky [1957], p. 21 or Kolchin [1948], p. 412.

Proposition 1. *Let k be a differential field of characteristic* 0 *with algebraically closed subfield of constants $\mathcal{C}$. Let $L(y) = 0$ be a linear differential equation over k. Then,*

1. *there exists a Picard-Vessiot extension for the equation,*
2. *any two Picard-Vessiot extensions for the equation are isomorphic.*

Let k be a differential field of characteristic 0 with algebraically closed subfield of constants $\mathcal{C}$. Let $L(y) = 0$ be a linear differential equation over k. Let K be the Picard-Vessiot extension of k for $L(x) = 0$, and write

$$V(L) = \{f \in K \mid L(f) = 0\}$$

for the space of solutions. $V(L)$ is generated as $\mathcal{C}$-vector space by n $\mathcal{C}$-linearly independent solutions $y_1, \ldots, y_n$. The **Galois group** of $L(y) = 0$, denoted by $Gal(K/k)$, is the differential Galois group $G(K/k)$ of the Picard-Vessiot extension K . A computational representation of $Gal(K/k)$ is obtained as follows:

Assume that $f \in V(L)$, then for any automorphism $\sigma \in Gal(K/k)$ we have

$$L(\sigma(f)) = \sigma(L(f)) = 0.$$

In other words, each automorphism moves a solution of $L(y) = 0$ to another solution. Consequently, $\sigma(f)$ is a linear $\mathcal{C}$-combination of the y_i's. This yields a matrix representation of $Gal(K/k)$.

Thus $Gal(K/k)$ acts faithfully on the vector space $V(L)$, and so $Gal(K/k)$ can be viewed as a subgroup of $\mathrm{GL}(V(L))$; more precisely, it is a linear algebraic group over $\mathcal{C}$. There is a Galois correspondence between algebraic subgroups of G and differential subfields of K. The fixed field of $Gal(K/k)$ under this correspondence is k.

A linear differential equation $L(y) = 0$ defined over k is said to be **solvable in terms of linear differential equations of lower order** if the associated Picard-Vessiot extension K of k lies in a tower of fields

$$k = k_0 \subset k_1 \subset \cdots \subset k_n,$$

where $k_i = k_{i-1}(t_i)$, and t_i is algebraic over k_{i-1}, or t_i satisfies a linear differential equation of lower order defined over k_{i-1}, for each i.

It has been shown in Singer [1989] that $L(y) = 0$ is solvable in terms of Liouvillian solutions (i.e., its Picard-Vessiot extension lies in a Liouvillian extension of k) if and only if its Galois group $Gal(K/k)$ contains a solvable (in the algebraic sense) subgroup of finite index. However, finding Liouvillian solutions is still hard and one attempt is to find these solutions by effectively searching over a bounded space (see Singer [1981]).

Notation 1. *For an order n monic scalar linear homogeneous differential equation*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

it is convenient to write

$$L(y) = \frac{w(y, y_1, \ldots, y_n)}{w(y_1, \ldots, y_n)} = 0$$

where $w = w(y_1, \ldots, y_n) = w_n$ is the Wronskian of $y_1, \ldots, y_n$, and w_{n-r} is obtained from w by replacing $y_1^{(n-r)}$ by $y_1^{(n)}$, $y_2^{(n-r)}$ by $y_2^{(n)}$ and so on (these determinants were called afterwards the ***generalized Wronskians****). Then*

$$a_{n-r} = -\frac{w_{n-r}}{w} \tag{3.1.3}$$

This means that the logarithmic derivative of each non-zero a_i can be expressed as a quotient of two w. In particular,

$$a_{n-1} = -\frac{w'}{w}, \text{ or } w' + a_{n-1}w = 0 \tag{3.1.4}$$

which is known as ***Liouville's relation****.*

Example 4. *Let $L_3(y) = 0$ be the third order monic scalar linear differential equation given by*

$$L_3(y) = y^{(3)} + a_2 y^{(2)} + a_1 y' + a_0 y = 0$$

with $a_0, a_1, a_2 \in k$. Let $\{y_1, y_2, y_3\}$ be a fundamental set of solutions of $L_3(y) = 0$. Then,

$$w(y, y_1, y_2, y_3) = \begin{vmatrix} y & y_1 & y_2 & y_3 \\ y' & y_1' & y_2' & y_3' \\ y^{(2)} & y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y^{(3)} & y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix} =$$

$$-\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix} y^{(3)} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix} y^{(2)} - \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix} y'$$

$$+ \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix} y,$$

where

$$w = w(y_1, y_2, y_3) = -\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix} = w_3$$

$$w_2 = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix} = w', \ w_1 = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}, \text{ and}$$

$$w_0 = \begin{vmatrix} y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}.$$

In other words,

$$L_3(y) = 0 \ \Rightarrow \ \frac{w(y, y_1, y_2, y_3)}{w(y_1, y_2, y_3)} = 0 \ \Rightarrow$$

$$y^{(3)} - \frac{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}} y^{(2)} - \frac{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}} y' - \frac{\begin{vmatrix} y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}} y = 0.$$

Therefore,

$$a_2 = -\frac{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}} = \frac{w'}{w}, \quad a_1 = -\frac{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}} = \frac{w_1}{w}, \text{ and}$$

$$a_0 = -\frac{\begin{vmatrix} y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}} = \frac{w_0}{w}$$

which correspond to Equation (3.1.3).

Let $(k, ')$ be a differential field such that its subfield of constants $\mathcal{C}$ is different from k and has characteristic 0, and let us consider the non-commutative ring $\mathcal{D} := k[\partial]$ of linear differential operators with coefficients in k. Let $L \in \mathcal{D}$ be an operator given by

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0.$$

Let us consider the scalar equation $L(y) = 0$, i.e.,

$$y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1 y' + a_0 y = 0.$$

If $y \in k$ is a solution of the equation $L(y) = 0$ such that $y' = u \in k$ then we will write, as a formal notation

$$y = \int u.$$

Now the homogeneous differential equation $L(y) = 0$ with fundamental solutions $y_1, \ldots, y_n$ is obtained by eliminating the n arbitrary constants c_i from the $n+1$ equations

$$\begin{aligned} y &= c_1 y_1 + \cdots + c_n y_n \\ y' &= c_1 y_1' + \cdots + c_n y_n' \\ &\vdots \\ y^{(n)} &= c_1 y_1^{(n)} + \cdots + c_n y_n^{(n)} \end{aligned}$$

and is therefore

$$w = w(y, y_1, \ldots, y_n) = 0,$$

where w is the Wronskian of $y, y_1, \ldots, y_n$. In its development, the coefficients of $y^{(n)}$ will be the $w(y, y_1, \ldots, y_n)$ which is not zero since $y_1, \ldots, y_n$ form a fundamental set.

An operator L is said to be **reducible** if there exists operators L_1 and L_2 of lower order such that $L = L_2 L_1$, in this case we say that L_1 is a right factor and L_2 is a left factor of L. If an operator is not reducible then it is called **irreducible**.

Remark 3.1.1. *If we interpret this definition in terms of the scalar equation we can say, in the old fashion way, that:*

"A linear homogeneous differential equation is called irreducible when it is has no common solutions with any other linear homogeneous differential equation of inferior order with the same kind of coefficients."

The next theorem describes a full factorization of L in terms of a fundamental system.

Theorem 6. *Let*

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0$$

be a linear differential operator of order n with coefficients in k. Let

$$\{y_1, y_2, \ldots, y_n\}$$

be a fundamental set of solutions for the scalar equation $L(y) = 0$, and define

$$\omega_0 = 1 \text{ and}$$

$$\omega_r := w(y_1, y_2, \ldots, y_r) \text{ for } r = 1, 2, \ldots, n.$$

Then

$$L(y) = (-1)^n \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(\frac{\omega_1^2}{\omega_2 \omega_0} \left(\frac{y}{\omega_1} \right)' \right)' \right)' \cdots \right)'$$

and

$$L = (-1)^n L_n L_{n-1} \ldots L_2 L_1$$

where

$$L_i = \partial - \alpha_i \text{ and } \alpha_i = \frac{\left(\frac{\omega_i}{\omega_{i-1}}\right)'}{\left(\frac{\omega_i}{\omega_{i-1}}\right)} \text{ for all } i = 1, \ldots, n.$$

Proof. Let us define

$$\Delta_0 = y \text{ and } \Delta_r = w(y, y_1, \ldots, y_r), \text{ for } r = 1, \ldots, n.$$

Using the formula

$$\Delta_r \omega_{r-1} = \Delta_{r-1}\omega_r' - \Delta_{r-1}'\omega_r$$

which is proved by partially expanding the determinants, we get

$$L(y) = \frac{\Delta_n}{\omega_n} = \frac{\omega_n \Delta_n \omega_{n-1}}{\omega_{n-1}\omega_n^2} = -\frac{\omega_n}{\omega_{n-1}} \left(\frac{\Delta_{n-1}}{\omega_n} \right)'.$$

Suppose, by induction hypothesis, that

$$L(y) = (-1)^{n-1} \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(\frac{\Delta_1}{\omega_2} \right)' \right)' \cdots \right)'$$

then

$$L(y) = (-1)^{n-1} \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(\frac{\omega_1^2 \Delta_1 \omega_0}{\omega_2 \omega_0 \omega_1^2} \right)' \right)' \cdots \right)' =$$

$$(-1)^{n-1} \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(\frac{\omega_1^2 (\Delta_0 \omega_1' - \Delta_0' \omega_1)}{\omega_2 \omega_0 \omega_1^2} \right)' \right)' \cdots \right)' =$$

$$(-1)^{n-1} \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(-\frac{\omega_1^2}{\omega_2 \omega_0} \left(\frac{\Delta_0' \omega_1 - \Delta_0 \omega_1'}{\omega_1^2} \right) \right)' \right)' \cdots \right)' =$$

$$(-1)^n \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(\frac{\omega_1^2}{\omega_2 \omega_0} \left(\frac{\Delta_0}{\omega_1} \right)' \right)' \right)' \cdots \right)'.$$

Therefore,

$$L(y) = (-1)^n \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}^2}{\omega_n \omega_{n-2}} \cdots \left(\frac{\omega_2^2}{\omega_3 \omega_1} \left(\frac{\omega_1^2}{\omega_2 \omega_0} \left(\frac{y}{\omega_1} \right)' \right)' \right)' \cdots \right)'$$

and using the fact that $\omega_0 = 1$ by definition, we obtain

$$L(y) = (-1)^n \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}}{\omega_n} \frac{\omega_{n-1}}{\omega_{n-2}} \cdots \left(\frac{\omega_2}{\omega_3} \frac{\omega_2}{\omega_1} \left(\frac{\omega_1}{\omega_2} \omega_1 \left(\frac{y}{\omega_1} \right)' \right)' \right)' \cdots \right)'.$$

Now, let us define

$$\alpha_i = \frac{\left(\frac{\omega_i}{\omega_{i-1}} \right)'}{\left(\frac{\omega_i}{\omega_{i-1}} \right)} \text{ for all } i = 1, \dots, n;$$

in particular $\alpha_1 = \omega_1'/\omega_1$. By the formula

$$a \left(\frac{z}{a} \right)' = z' - \frac{a'}{a} z \text{ for all } a, z \in k,$$

we have

$$\omega_1 \left(\frac{y}{\omega_1} \right)' = y' - \frac{\omega_1'}{\omega_1} y = y' - \alpha_1 y = (\partial - \alpha_1)(y)$$

and taking $z_1 = y' - \alpha_1 y$ we get

$$\frac{\omega_2}{\omega_1} \left(\frac{\omega_1}{\omega_2} z_1 \right)' = \frac{\omega_2}{\omega_1} \left(\frac{z_1}{\frac{\omega_2}{\omega_1}} \right)' = z_1' - \frac{\left(\frac{\omega_2}{\omega_1} \right)'}{\left(\frac{\omega_2}{\omega_1} \right)} z_1 = z_1' - \alpha_2 z_1 = (\partial - \alpha_2)(z_1) =$$

$$(\partial - \alpha_2)(y' - \alpha_1 y) = (\partial - \alpha_2)(\partial - \alpha_1)(y).$$

Suppose, by induction hypothesis, that

$$\frac{\omega_{n-1}}{\omega_{n-2}} \left(\frac{\omega_{n-2}}{\omega_{n-1}} \frac{\omega_{n-2}}{\omega_{n-3}} \cdots \left(\frac{\omega_2}{\omega_3} \frac{\omega_2}{\omega_1} \left(\frac{\omega_1}{\omega_2} \omega_1 \left(\frac{y}{\omega_1} \right)' \right)' \right)' \cdots \right)' =$$

$$(\partial - \alpha_{n-1}) \dots (\partial - \alpha_2)(\partial - \alpha_1)(y)$$

and put

$$z_{n-1} = (\partial - \alpha_{n-1}) \dots (\partial - \alpha_2)(\partial - \alpha_1)(y)$$

then

$$L(y) = (-1)^n \frac{\omega_n}{\omega_{n-1}} \left(\frac{\omega_{n-1}}{\omega_n} z_{n-1} \right)' = (-1)^n \frac{\omega_n}{\omega_{n-1}} \left(\frac{z_{n-1}}{\frac{\omega_n}{\omega_{n-1}}} \right)' =$$

$$(-1)^n \left(z_{n-1}' - \frac{\left(\frac{\omega_n}{\omega_{n-1}} \right)'}{\left(\frac{\omega_n}{\omega_{n-1}} \right)} z_{n-1} \right) = (-1)^n (z_{n-1}' - \alpha_n z_{n-1}) =$$

$$(-1)^n (\partial - \alpha_n)(z_{n-1}) = (-1)^n (\partial - \alpha_n)(\partial - \alpha_{n-1}) \dots (\partial - \alpha_2)(\partial - \alpha_1)(y).$$

Therefore,

$$L = (-1)^n (\partial - \alpha_n)(\partial - \alpha_{n-1}) \cdots (\partial - \alpha_3)(\partial - \alpha_2)(\partial - \alpha_1).$$

□

Remark 3.1.2. *Note that the order of the factors $(\partial - \alpha_i)$ must in general be preserved, for it is not true for any two suffixes i and j*

$$(\partial - \alpha_i)(\partial - \alpha_j) = (\partial - \alpha_j)(\partial - \alpha_i).$$

In other words, the factors of the differential operator are in general not permutable (Landau [1902]).

3.2 Rational Solutions of Linear Differential Equations

In this section we present a method for finding rational solutions of linear differential equations with coefficients in $k(x)$ in case $\mathrm{char}(k) = 0$ and ordinary derivative d/dx. This method generalizes the well known Frobenius method for solving second-order ordinary differential equations relative to a singular point, like e.g. the Euler equation, the hypergeometric equation or equations of Fuchsian type. See for instance Ince [1964]), Cohen, Cuypers, and Sterk [1999] (for equations of third order) and van der Put and Singer [2003]. It will appear again later in connection with finding exponential solutions of linear differential equations. Although the coefficients of the linear differential equations that we consider here are in $k(x)$, we will work in the field $k((x))$ of formal Laurent series. In the sequel we denote the algebraic closure of an arbitrary field K by $\overline{K}$.

Let k be a field of characteristic 0 and $k[[x]]$ the ring of formal power series. A typical element of $k[[x]]$ is

$$\sum_{i=0}^{\infty} a_i x^i, \text{ where } a_i \in k.$$

The quotient field of $k[[x]]$, denoted by $k((x))$, is the field of formal Laurent series. $\overline{k((x))}$ is contained in the algebraically closed field of formal **Puiseux series**

$$\bigcup_{n \in \mathbb{N}} \overline{k}((x^{1/n})),$$

A typical non-zero element $a \in k((x))$ can be written as

$$a = x^m \sum_{j \geq 0} a_j x^j \text{ where } a_0 \neq 0 \text{ and } m \in \mathbb{Z}.$$

The order $\mathrm{ord}(a)$ of a is the exponent m of the first non-vanishing term of a. By definition $\mathrm{ord}(0) = \infty$. In Chapter 4 we present more details on Puiseux series.

We consider $k(x)$ and $k((x))$ as differential fields equipped with derivation $\partial = d/dx$.

Let $L \in k(x)[\partial]$ be the operator

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0.$$

We write the coefficients of L as partial fractions

$$a_i = \sum_{j=0}^{d_i} p_{i,j} x^j + \sum_{\sigma=1}^{s} \sum_{j=d_i^\sigma}^{-1} q_{i,j}^\sigma (x - x_\sigma)^j + \sum_{\tau=1}^{t} \sum_{j=1}^{e_i^\tau} \frac{B_{i,j}^\tau(x)}{A^\tau(x)^j}, \text{ for all } i.$$

An element $a \in k((x))$ is said to have a **pole of order** n **at** $x = x_0$, if in the Laurent series, $a_m = 0$ for $m < -n$ and $a_n \neq 0$, i.e.,

$$a = \sum_{i \geq n} a_i (x - x_0)^i.$$

a is said to have a **pole of order** n **at** ∞, if the Laurent series of $a \in k((x^{-1}))$ at 0 has only finitely many negative degree terms, starting with $-n$.

Proposition 2. *A solution f of the equation*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$$

can only have a pole at α if at least one of the a_i has a pole at α. Also, ∞ is a possible pole of f. Hence the location of the possible poles of f is known.

Proof. Suppose that f has a pole at 0 of order s. Then the expansion of f at 0 is

$$x^s + l_{s+1}x^{s+1} + \cdots \text{ with } s < 0.$$

If $a_i(0) \neq 0$ for all i, then the lowest power of x in the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

is $s - n$ and it would appear in only one member, namely the first one $y^{(n)}$ and as a consequence it must have 0 as a coefficient, i.e.,

$$s(s-1)(s-2)\ldots(s-n+1) = 0.$$

Then, $s = 0$, $s = 1$, ... , $s = n - 1$. In any case, $s \geq 0$ which is a contradiction.

□

If it exists, a rational solution of the scalar equation $L(y) = 0$ has the form

$$f = \sum_{j=0}^{M} p_j x^j + \sum_{\sigma=1}^{s} \sum_{j=M^\sigma}^{-1} q_j^\sigma (x - x_\sigma)^j + \sum_{\tau=1}^{t} \sum_{j=1}^{N^\tau} \frac{B_j^\tau(x)}{A^\tau(x)^j} + \sum_{i=1}^{N} \frac{\gamma_i}{x - c_i}.$$

Let $p \in k((x))$ be given by

$$p = x^m \sum_{i \geq 0} p_i x^i, \text{ where } p_0 \neq 0 \text{ and } m \in \mathbb{Z}.$$

The **indicial polynomial** $I_p(m)$ in m (seen as variable) of degree $\leq n$, for the linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$$

at 0 is the coefficient of the term of lowest degree of x after substituting p in the differential equation. It is obvious that $I_p(m)$ will be independent of the coefficients $a_1, a_2, \ldots$, and will involve a_0 as a multiplicative factor. The roots of $I_p(m)$ (in an algebraic closure of k) are called local exponents of the equation at 0. The equation $I_p(m) = 0$ is called the **indicial equation**.

Now we are ready to present the algorithm.

Algorithm:

Let us consider the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0 \tag{3.2.5}$$

where some $a_i \in k(x)$ has a pole at 0. Suppose that the expansion of f in the Laurent series at 0 is

$$f = \frac{u}{v} = x^s + l_{s+1}x^{s+1} + \cdots \text{ with } s \in \mathbb{Z} \text{ and } s < 0.$$

The expansion of the a_i in Laurent series at 0 is written as

$$a_i = \sum_{j \geq n_i} a_{i,j} x^j \text{ for all } i,$$

where $a_{i,n_i} \in k^*$ and $n_i \in \mathbb{Z}$. If $a_i = 0$ we put $n_i = \infty$ and a_{i,n_i} has no meaning. Then the lowest power of x in Equation (3.2.5) are

$$s(s-1)(s-2)\dots(s-n+1)x^{s-n},$$

$$s(s-1)(s-2)\dots(s-n+1)(s-n+2)a_{n-1,n_{n-1}}x^{s-n+1+n_{n-1}},$$

$$\vdots$$

$$sa_{1,n_1}x^{s-1+n_1}, \text{ and}$$

$$a_{0,n_0}x^{s+n_0}.$$

Let m denote the minimum of $\{-n, -n+1+n_{n-1}, \dots, -1+n_1, n_0\}$. The coefficient $I(s)$ of x^{s+m} can be written as

$$\epsilon_1 s(s-1)(s-2)\dots(s-n+1)+$$

$$+\epsilon_2 s(s-1)(s-2)\dots(s-n+1)(s-n+2)a_{n-1,n_{n-1}} + \dots +$$

$$\epsilon_1 s a_{1,n_1} + \epsilon_0 a_{0,n_0},$$

where $\epsilon_i = 1$ if the corresponding element in

$$\{-n, -n+1+n_{n-1}, \dots, -1+n_1, n_0\}$$

is minimal and $\epsilon_i = 0$ otherwise.

The expression for I is a non-zero polynomial in s (seen as a variable) of degree $\leq n$, this is precisely the indicial polynomial for Equation (3.2.5) at 0.

Since f satisfies Equation (3.2.5) the coefficient $I(s)$ of x^{s+m} must be 0. Thus s is a solution of the equation

$$I(s) = 0$$

which is exactly the indicial equation for Equation (3.2.5) at 0. We have three possibilities, namely

1. If there is no s with $I(s) = 0$ then we can stop the calculations since in that case there is no rational solution $f \neq 0$ of the equation.
2. If there is no negative integer s with $I(s) = 0$ but there is an integer $r \geq 0$ then we define $s_0 = 0$.
3. If there is a negative integer solution of $I(s) = 0$ then $s_0 < 0$ denotes the smallest one.

In other words, the possible values s for the exact power x^s are the negative integers s with $I(s) = 0$.

Now we perform a similar calculation at ∞. Let $\text{ord}_s(f)$ denote the order of the function f at the point s.

Let α_i denote the poles of a_i. For every i, the method above yields an integer $s_i \leq 0$ such that, for any rational solution $f \neq 0$, one has $\text{ord}_{\alpha_i}(f) \geq s_i$, or there are no rational solutions.

This means that (working in the Laurent series field $k((x^{-1}))$ we can write

$$f = \frac{T}{N} \in k(x)^*$$

with known

$$N = \prod_i (x - \alpha_i)^{-s_i} \text{ and with some polynomial } T.$$

In order to estimate the degree of T let us consider the expansions in Laurent series of f and the a_i at ∞ (i.e., we make a calculation in the differential field $k((x^{-1})))$. Suppose that the expansions have the form

$$f = x^t + g_{t+1}x^{t+1} + \cdots \text{ and}$$
$$a_i = \sum_{j \geq m_i} b_{i,j}x^j \text{ for all } i.$$

If $a_i \neq 0$, then $b_{i,m_i} \in k^*$ and $m_i \in \mathbb{Z}$. If $a_i = 0$ then we put $m_i = -\infty$ and b_{i,m_i} has no meaning. Then, the possible highest power of x in Equation (3.2.5) are

$$t(t+1)(t+2)\dots(t+n+1)x^{t-n},$$

$$t(t+1)(t+2)\dots(t+n+1)(t+n+2)b_{n-1,m_{n-1}}x^{t-n+1+m_{n-1}},$$

$$\vdots$$

$$tb_{1,m_1}x^{t-1+m_1}, \text{ and}$$

$$b_{0,m_0}x^{t+m_0}.$$

Let M denote the maximum of $\{-n, -n+1+m_{n-1}, \dots, -1+m_1, m_0\}$. Let $J(t)$ be the expression

$$\epsilon_1 t(t+1)(t+2)\dots(t+n+1)+$$

$$\epsilon_2 t(t+1)(t+2)\dots(t+n+1)(t+n+2)b_{n-1,m_{n-1}} + \cdots +$$

$$\epsilon_1 tb_{1,m_1} + \epsilon_0 b_{0,m_0},$$

where $\epsilon_i = 1$ if the corresponding term is equal to M and $\epsilon_i = 0$ otherwise.

Then $J(t)$ is a non-zero polynomial of degree $\leq n$ in t (seen as a variable). This is the indicial polynomial of Equation (3.2.5) at ∞.

If there is no integer t with $J(t) = 0$ then we stop the procedure. In other case, let s_∞ denote the largest integer that is a zero in J. Then we find that $t \leq s_\infty$.

Expanding $f = \dfrac{T}{N}$ at ∞ leads to the inequality

$$\deg(T) \leq s_\infty + \deg(N).$$

Let d be the bound for the degree of T and write

$$T = t_d x^d + \cdots + t_0.$$

The equation satisfied by f provides an order n equation for T. This leads to a set of linear equations for the coefficients t_i. With linear algebra one can find all solutions. This ends the algorithm.

Example 5. *Consider the equation*

$$y^{(3)} - \frac{8x^2 - 63x - 27}{(24x+27)x}y^{(2)} + \frac{448x^2 + 1080x + 1080}{3(8x+9)^2 x}y^{(1)} - \frac{24}{(8x+9)^2 x}y = 0$$

with coefficients in partial fraction decomposition

$$a_2 = -\frac{1}{3} + \frac{16}{8x+9} + \frac{1}{x}, \; a_1 = -\frac{152}{9(8x+9)} - \frac{128}{(8x+9)^2} + \frac{40}{9x}, \; \textit{and}$$

$$a_0 = \frac{64}{27(8x+9)} + \frac{64}{3(8x+9)^2} - \frac{8}{27x}.$$

The poles of the coefficients are at 0 *and at* $-9/8$. *First, let us consider the expansion in Laurent series of the* a_i *at* 0, *that is*

$$a_2 = x^{-1} + \frac{13}{9} - \frac{128}{81}x + \frac{1024}{729}x^2 - \frac{8192}{6561}x^3 + \frac{65536}{59049}x^4 + O(x^5),$$

$$a_1 = \frac{40}{9}x^{-1} - \frac{280}{81} + \frac{1088}{243}x - \frac{34304}{6561}x^2 + \frac{339968}{59049}x^3 - \frac{360448}{59049}x^4 + O(x^5),$$

$$a_0 = -\frac{8}{27}x^{-1} + \frac{128}{243} - \frac{512}{729}x + \frac{16384}{19683}x^2 - \frac{163840}{177147}x^3 + \frac{524288}{531441}x^4 + O(x^5).$$

Then the indicial polynomial at 0 *is*

$$I(s) = \epsilon_1 s(s-1)(s-2) + \epsilon_2(s-1)a_{2,n_2} + \epsilon_3 s a_{1,n_1} + \epsilon_0 a_{0,n_0}$$

where $\epsilon_i = 1$ *if the corresponding element in* $\{-3, -2+n_2, -1+n_1, n_0\}$ *is minimal and* $\epsilon_i = 0$ *otherwise. Since* $n_2 = n_1 = n_0 = -1$ *we have*

$$I(s) = s^3 - 3s^2 + 2s + (s^2 - s)a_{2,n_2} = s^3 - 2s^2 + s.$$

Therefore, the indicial equation at 0 *is*

$$I(s) = 0, \text{ i.e., } s^3 - 2s^2 + s = 0$$

which has solutions $s = 0, 1, 1$, *whence* $s_0 = 0$. *Then a term of the possible rational solution is*

$$f_1(x) = c_1 x^0 = c_1, \text{ with } c_1 \text{ a constant.}$$

Now, let us consider the expansion in Laurent series of the a_i *at* $-9/8$, *that is*

$$a_2 = 2\left(x+\frac{9}{8}\right)^{-1} - \frac{11}{9} - \frac{64}{81}\left(x+\frac{9}{8}\right) - \frac{512}{729}\left(x+\frac{9}{8}\right)^2 -$$

$$\frac{4096}{6561}\left(x+\frac{9}{8}\right)^3 - \frac{32768}{59049}\left(x+\frac{9}{8}\right)^4 - \frac{262144}{531441}\left(x+\frac{9}{8}\right)^5 + O\left(\left(x+\frac{9}{8}\right)^6\right),$$

$$a_1 = -2\left(x+\frac{9}{8}\right)^{-2} - \frac{19}{9}\left(x+\frac{9}{8}\right)^{(-1)} - \frac{320}{81} - \frac{2560}{729}\left(x+\frac{9}{8}\right) -$$

$$\frac{20480}{6561}\left(x+\frac{9}{8}\right)^2 - \frac{163840}{59049}\left(x+\frac{9}{8}\right)^3 - \frac{1310720}{531441}\left(x+\frac{9}{8}\right)^4 -$$

$$\frac{10485760}{4782969}\left(x+\frac{9}{8}\right)^5 + O\left(\left(x+\frac{9}{8}\right)^6\right),$$

$$a_0 = \frac{1}{3}\left(x+\frac{9}{8}\right)^{-2} + \frac{8}{27}\left(x+\frac{9}{8}\right)^{-1} + \frac{64}{243} + \frac{512}{2187}\left(x+\frac{9}{8}\right) +$$

$$\frac{4096}{19683}\left(x+\frac{9}{8}\right)^2 + \frac{32768}{177147}\left(x+\frac{9}{8}\right)^3 + \frac{262144}{1594323}\left(x+\frac{9}{8}\right)^4 +$$

$$\frac{2097152}{14348907}\left(x+\frac{9}{8}\right)^5 + O\left(\left(x+\frac{9}{8}\right)^6\right).$$

Then, the indicial polynomial at $-9/8$ *is*

$$I(t) = \epsilon_1 t(t-1)(t-2) + \epsilon_2(t-1)b_{2,m_2} + \epsilon_3 t b_{1,m_1} + \epsilon_0 b_{0,m_0}$$

where $\epsilon_i = 1$ *if the corresponding element in* $\{-3, -2 + m_2, -1 + m_1, m_0\}$ *is minimal and* $\epsilon_i = 0$ *otherwise. Since* $m_2 = -1$, $m_1 = -2$ *and* $m_0 = -2$ *we have*

$$I(t) = t^3 - 3t^2 + 2t + (t^2 - t)b_{2,m_2} + tb_{1,m_1} = t^3 - 3t^2 + 2t + 2t^2 - 2t - 2t.$$

Therefore, the indicial equation at $-9/8$ *is*

$$I(t) = 0, \text{ i.e., } t^3 - t^2 - 2t = 0$$

which has solutions $t = 0, -1, 2$, *whence* $t_{9/8} = -1$. *Then, another term of the possible rational solution is*

$$f_2 = \frac{c_2}{x + \frac{9}{8}} \text{ with } c_1 \text{ a constant.}$$

After substituting

$$f_1 + f_2 = c_1 + \frac{c_2}{x + \frac{9}{8}}$$

in the equation we obtain

$$-\frac{24(b_1 + 8b_2)}{(8x + 9)^2 x} = 0 \Rightarrow c_1 = -8c_2.$$

Taking $c_2 = -\dfrac{1}{64}$ *we get* $c_1 = \dfrac{1}{8}$, *and in fact the rational solution is*

$$f = \frac{1 + x}{8x + 9}.$$

If we consider the expansion in Laurent series of the a_i *at* ∞ *we obtain the same result.*

3.3 Exponential Solutions of Linear Differential Equations

A major subproblem for algorithms that either factor ordinary linear differential equations or compute their closed form solutions is to find solutions y which satisfy $y'/y \in \overline{k}$, where k is the constant field for the coefficients of the equations, these solutions are called exponentials. In other to look for exponential solutions in $\overline{k}$ one should consider the associated Riccati equation and search for its rational solutions, because the key property of the Riccati equation is:

If $y \neq 0$ is any solution of a linear differential equation, then $u = y'/y$ a solution of the corresponding Riccati equation, and vice versa.

Although the exponential solutions form only a subspace of the Liouvillian solutions, the main subalgorithm of the algorithms that find the Liouvillian solutions Kovacic [1986] and Singer [1991] is to find the exponential solution of higher order equations.

There are several algorithms for finding rational solutions of the Riccati equations in particular cases, among them we have:

- solve_riccati from Bronstein [1992a], coefficients in $k(x)$ and solutions over $\overline{k}(x)$.
- RiccatiRational from Schwarz [1994], coefficients in $\mathbb{Q}(x)$ and solutions over $\mathbb{Q}(x)$.
- An improved solve_riccati from Li and Schwarz [2001], coefficients in $k(x)$ and solutions over $k(x)$.

There is however no known general algorithm for finding rational solutions of Riccati equations.

In this section we present the RiccatiRational algorithm which finds solutions in $k(x)$ of the Riccati equation associated with a linear differential equation having coefficients from $k(x)$. The algorithm looks for bounds on the coefficients of possible solutions and reduces the problem to solving a linear system. If this system is feasible we obtain a rational solution of the associated Riccati equation and at once a right-hand factor of the original linear differential equation.

Let k be a differential field of characteristic 0 with algebraically closed field of constants $\mathcal{C}$. Let $L \in k[\partial]$ be given by

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0.$$

Let $K \supset k$ be the Picard-Vessiot extension for $L(y) = 0$, set

$$V := \{\, y \in K \mid L(y) = 0. \}$$

and let $G \subset GL(V)$ be the differential Galois group of L.

A non-zero element $y \in V \subset K$ with $L(y) = 0$ is called an **exponential solution** of L if

$$u := \frac{y'}{y}$$

lies in k. We will write, as a formal notation,

$$y = e^{\int u}.$$

Let $y, u \in K$ satisfy $y' = uy$. Formally differentiating this identity yields

$$y^{(i)} = P_i(u, u', \ldots, u^{(i-1)})y,$$

where the P_i are polynomials with integer coefficients satisfying

$$P_0 = 1, \text{ and } P_i = P'_{i-1} + uP_{i-1} \text{ for all } i \geq 1.$$

Furthermore, $y \neq 0$ satisfies $L(y) = 0$ if and only if $u := \dfrac{y'}{y}$ satisfies

$$R(u) = P_n\left(u, \ldots, u^{(n-1)}\right) + a_{n-1}P_{n-1}\left(u, \ldots, u^{(n-2)}\right) + \ldots + a_0 = 0. \tag{3.3.6}$$

Equation (3.3.6) is the **Riccati equation associated** with $L(y) = 0$.

The next proposition proposed as an exercise in van der Put and Singer [2003], provides the relation between exponential solutions of a linear differential equation and rational solutions of the associated Riccati equation. Furthermore it points the way to detect first order right-hand factors.

Proposition 3. *1. An element $v \in k$ is a solution of the Riccati equation if and only if $\partial - v$ is a right-hand factor of L (i. e., $L = \tilde{L}(\partial - v)$ for some $\tilde{L}$).*

2. The element $v \in K$ is a solution of the Riccati equation if and only if there is a $y \in V \subset K$, $y \neq 0$ with $y'/y = v$.

Proof. 1. If $v \in k$ is a solution of the Riccati equation $R(u) = 0$ there exist $y_1 \in K$ with $y_1 \neq 0$ and $y_1' = vy_1$ such that y_1 is a solution of the linear equation $L(y) = 0$. Let L_1 be the operator defined by

$$L_1 = \partial - v,$$

where its scalar equation is

$$L_1(y) = 0 \;\Rightarrow\; (\partial - v)(y) = 0 \;\Rightarrow\; \partial(y) - vy = 0 \;\Rightarrow\; y' - \frac{y_1'}{y_1}y = 0.$$

Then y_1 is also a solution of the equation $L_1(y) = 0$, by Theorem (6), $\partial - v$ is a right-hand factor of L.

Conversely, suppose that $v \in k$ is such that $\partial - v$ is a right-hand factor of $L(y) = 0$. Let $y_0 \in K$ be solution of the equation $\partial - v$, that is

$$(\partial - v)(y_0) = 0 \;\Rightarrow\; \partial(y_0) - vy_0 = 0 \;\Rightarrow y_0' - vy_0 = 0 \; y_0' = vy_0.$$

Since $\partial - v$ is a right-hand factor of L and $y_0 \in K$ is a solution of the equation $y' - vy = 0$ we have $L(y_0) = 0$ and so $y_0 \in V$.

Now, $y_0 \in V \subset K$ and $y_0' = vy_0$ with $v \in K$, by the previous considerations we obtain that v is a solution of the Riccati associated equation of $L(y) = 0$.

2. If the element $v \in K$ is a solution of the Riccati equation, then by part 1. we have that $\partial - v$ is a right-hand factor of L, and so $\partial - v$ and L have a common solution. Let $y \in V$ with $y \neq 0$ such that $(\partial - v)(y) = 0$. Whence, we get

$$\frac{y'}{y} = v.$$

Conversely, if $y_1 \in V \subset K$ is such that $y_1 \neq 0$ and $y_1'/y_1 = v_1$ with $v_1 \in K$ then $L(y_1) = 0$. On the other hand, $y_1' = v_1 y_1$ and by the previous considerations we obtain that $v \in K$ is a solution of the Riccati equation.

□

An element $v \in K$ is called a Riccati solution for the equation $L(y) = 0$ if v is the logarithmic derivative $v = y'/y$ of some non-zero solution of y of $L(y) = 0$. If v is an algebraic Riccati solution (a Riccati solution in $\overline{k}$) then the minimal polynomial of v over k is called a Riccati polynomial of L.

Since for an exponential solution y of $L(y) = 0$, $u = y'/y \in \overline{k}(x)$, finding exponential solutions of $L(y) = 0$ is equivalent to finding the rational solutions of 3.3.6.

Next we present the RiccatiRational algorithm from Schwarz [1994].

Algorithm RiccatiRational Schwarz [1994]:

Input: $R(u) = 0$ with $a_i \in \mathbb{Q}(x)$ given as partial fractions

$$a_i = \sum_{j=0}^{d_i} p_{i,j} x^j + \sum_{\sigma=1}^{s} \sum_{j=d_i^\sigma}^{-1} q_{i,j}^\sigma (x - x_\sigma)^j + \sum_{\tau=1}^{t} \sum_{j=1}^{e_i^\tau} \frac{B_{i,j}^\tau(x)}{A^\tau(x)^j}$$

for $i = 0, \ldots, n-1$.

Output: A rational solution of the form:

$$q = \sum_{j=0}^{M} p_j x^j + \sum_{\sigma=1}^{s} \sum_{j=M^\sigma}^{-1} q_j^\sigma (x - x_\sigma)^j + \sum_{\tau}^{t} \sum_{j=1}^{N^\tau} \frac{B_j^\tau(x)}{A^\tau(x)^j} + \sum_{i=1}^{N} \frac{\gamma_i}{x - c_i}.$$

with $\gamma_i \in \mathbb{N}$, is the well known structure of rational solutions to be determined.

1. Determine the bounds M, M^σ, and N^τ.
2. Determine the algebraic systems for the coefficients $p_0, \ldots, p_M, q_1^\sigma, \ldots, q_{M^\sigma}^\sigma$, and $b_{1,0}^\tau, \ldots, b_{N^\tau, m^\tau - 1}^\tau$, solve them and construct the solution candidates from these solutions.
3. For each candidate found in (2.), determine the equation for the possible polynomial factor and determine a bound for it.
4. Determine the polynomial factor and return the complete solution.

The next example is taken from Schwarz [1994], however we make the computations differently. Instead to compute the bounds M, M^σ, and N^τ with the theorems Bound 1, 2 and 3 of Schwarz [1989], in an easier way, we compute the indicial equations of the polynomial part, rational part at any pole and the logarithmic derivative part at any other kind of singularities.

Example 6. *Consider the equation*

$$y^{(2)} + \left[x^2 + 4 + \frac{3}{(x-5)^2} + \frac{x+3}{(x^2+1)^2}\right] y^{(1)} +$$
$$\left[x^2 + 2x + 3 + \frac{3}{(x-5)^2} - \frac{6}{(x-5)^3} + \frac{x}{(x^2+1)^2} - \frac{12x-4}{(x^2+1)^3}\right] y = 0$$

where the indicial equation for the polynomial part is

$$P(m) = 0 \;\Rightarrow\; 1 = 0,$$

which has no solutions, then

$$M = \max\left(2,\ \textit{integer solution of } P(m) = 0\right) = 2$$

So the bound for the polynomial part is $M = 2$, that means that the possible polynomial part of the solution has degree 2, say

$$u_p = p_2 x^2 + p_1 x + p_0.$$

Substituting the possible polynomial part of the solution in the equation

$$u^{(1)} + u^2 + (x^2+4)u + x^2 + 2x + 3 = 0$$

we get

$$2p_2 + p_1 + (p_2x^2 + p_1x + p_0)^2 + (x^2+4)(p_2x^2 + p_1x + p_0) + x^2 + 2x + 3 = 0 \;\Rightarrow$$

$$(p_2^2 + p_2)x^4 + (2p_2p_1 + p_1)x^3 + (p_1^2 + 2p_2p_0 + 4p_2 + p_0 + 1)x^2 +$$
$$(2p_2 + 2p_1p_0 + 4p_1 + 2)x + (p_2^2 + p_1 + 4p_0 + 3) = 0 \;\Rightarrow$$

$$\begin{cases} p_2^2 + p_2 &= 0 \\ 2p_2p_1 + p_1 &= 0 \\ p_1^2 + 2p_2p_0 + 4p_2 + p_0 + 1 &= 0 \;\Rightarrow \\ 2p_2 + 2p_1p_0 + 4p_1 + 2 &= 0 \\ p_2^2 + p_1 + 4p_0 + 3 &= 0 \end{cases}$$

$$\begin{cases} (p_2+1)p_2 &= 0 \\ (p_2 + \frac{1}{2})p_1 &= 0 \\ p_2p_0 + 2p_2 + \frac{1}{2}p_1^2 + \frac{1}{2}p_0 + \frac{1}{2} &= 0 \end{cases}.$$

All coefficients are determined uniquely by this system, which has solution $p_2 = -1$, $p_1 = 0$ and $p_0 = -3$. Therefore, the polynomial part of the possible solution is

$$u_p = -x^2 - 3.$$

We try to find a possible bound at the rational pole $x_1 = 5$, where the indicial equation is

$$Q(m^\sigma) = 0 \;\Rightarrow\; 3m^\sigma - 6 = 0 \;\Rightarrow\; m^\sigma = 2$$

with

$$M^\sigma = \min\left(\,\textit{negative integer solution of} Q(m^\sigma) = 0\,\right) = -2.$$

So the bound for the rational part is $M^\sigma = -2$, that means that the possible rational part of the solution has the form

$$u_r = b_1(x-5)^{-1} + b_2(x-5)^{-2} = \frac{b_1}{x-5} + \frac{b_2}{(x-5)^2}.$$

Substituting the possible rational part of the solution in the equation

$$u^{(1)} + u^2 + \frac{3}{(x-5)^2}u + \frac{3}{(x-5)^2} - \frac{6}{(x-5)^3} = 0$$

we get

$$-\frac{b_1}{(x-5)^2} - \frac{2b_2}{(x-5)^3} + \frac{b_1^2}{(x-5)^2} + \frac{2b_1b_2}{(x-5)^3} + \frac{b_2^2}{(x-5)^4} +$$
$$\frac{3}{(x-5)^2}\left[\frac{b_1}{x-5} + \frac{b_2}{(x-5)^2}\right] + \frac{3}{(x-5)^2} - \frac{6}{(x-5)^3} = 0 \;\Rightarrow$$

$$\begin{cases} b_2^2 + 3 &= 0 \\ -2b_2 + 2b_1b_2 + 3b_1 - 6 &= 0 \\ -b_1 + b_1^2 + 3 = 0 \end{cases} \Rightarrow \begin{cases} (b_2+3)b_3 &= 0 \\ b_1b_2 - b_2 + \frac{3}{2}b_1 - 3 &= 0 \end{cases}.$$

The complete solution is obtained as $b_2 = -3$ and $b_1 = 0$. Therefore, the rational part of the possible solution is

$$u_r = -\frac{3}{(x-5)^2}.$$

Finally, we try to find a possible bound at the singularity x^2+1, where the indicial equation is

$$Q(n^\tau) = 0 \;\Rightarrow\; \left(-\frac{3}{4} - \frac{1}{4}i\right)n^\tau + \left(\frac{3}{2} + \frac{1}{2}i\right) = 0 \;\Rightarrow\; n^\tau = 2$$

with

$$N^\tau = \min\left(\,\textit{negative integer solution of} Q(n^\tau) = 0\,\right) = -2.$$

So the bound for the logarithmic derivative part is $N^\tau = -2$, that means that the possible logarithmic part of the solution has the form

$$u_l = (xb_2 + b_1)(x^2+1)^{-1} + (xb_4 + b_3)(x^2+1)^{-2} = \frac{xb_2 + b_1}{x^2+1} + \frac{xb_4 + b_3}{(x^2+1)^2}.$$

Substituting the possible logarithmic derivative part in the equation

$$u^{(1)} + u^2 + \frac{x+3}{(x^2+1)^2}u + \frac{x}{(x^2+1)^2} - \frac{12x-4}{(x^2+1)^3} = 0$$

we get

$$-\frac{b_2x^4+2b_1x^3+3b_4x^2+4b_3x+2b_1x-b_4-b_2}{(x^2-1)^3}+$$

$$u^2+\frac{x+3}{(x^2+1)^2}u+\frac{x}{(x^2+1)^2}-\frac{12x-4}{(x^2+1)^3}=0 \Rightarrow$$

$$\begin{cases} b_2^2-b_2 &= 0\\ 2b_1b_2-2b_1+1 &= 0\\ 2b_2b_4-3b_4+b_1^2+2b_2^2 &= 0\\ 2b_1b_4+2b_2b_3+4b_1b_2+3b_2-3b_1-10-4b_3 &= 0 \Rightarrow\\ 2b_1b_3+2b_2b_4+4+2b_2+3b_1-b_4+2b_1^2+b_2^2+b_4^2 &= 0\\ 2b_1b_4+2b_3b_4+2b_2b_3+2b_1b_2+3b_2+3b_4-11-b_1-3b_3 &= 0\\ 4+3b_1+b_2+3b_3+b_4+b_1^2+2b_1b_3+b_3^2 &= 0 \end{cases}$$

$$\begin{cases} b_1b_4+b_2b_3+\frac{1}{2}b_1+\frac{3}{2}b_2-2b_3-6 &= 0\\ b_4^2+2b_1b_3-2b_2b_4+5b_4-b_2+3b_1+4 &= 0\\ b_4^2+b_4-b_3^2-3b_3 &= 0\\ b_3b_4+\frac{3}{4}b_4+\frac{1}{2}b_3 &= 0 \end{cases}.$$

The non-linear system comprising the highest equation for b_3 and b_4 has to be transformed into a Gröbner bases. Due to factorization two non-trivial alternatives are obtained:

$$\begin{cases} b_3+3 &= 0\\ b_4+1 &= 0 \end{cases} \quad or \quad \begin{cases} b_4^2+3b_3+5 &= 0\\ b_3^2+3b_3+\frac{5}{2} &= 0 \end{cases}.$$

From the first alternative we get $b_4=-1$ and $b_3=-3$. Substituting them in the remaining equations, the linear system

$$\begin{cases} b_2+\frac{2}{3}b_1 &= 0\\ b_2-3b_1 &= 0 \end{cases} \Rightarrow \begin{cases} b_2 &= 0\\ b_1 &= 0 \end{cases} \Rightarrow$$

is obtained with the solution $b_1=b_2=0$.Therefore, the logarithmic derivative part of the possible solution is

$$u_l=-\frac{x+3}{(x^2+1)^2}.$$

Combining these results yields the solution candidate

$$u=u_p+u_r+u_l=-x^2-3-\frac{3}{(x-5)^2}-\frac{x+3}{(x^2+1)^2}.$$

It turns out that this is already a genuine solution of the Riccati equation. It generates the factorization

$$(\partial+1)\left[\partial+x^2+3+\frac{3}{(x-5)^2}+\frac{x+3}{(x^2+1)^2}\right].$$

The other two alternatives lead to complex expressions which do not turn into solutions of the Riccati equation.

3.4 The Beke's Algorithm

In this section we will give an exposition of the classical Beke algorithm for factoring linear differential operators in the field $k(x)$. The original paper Beke [1894] appeared at the end of the 19th century and after almost one hundred years it has been improved and implemented in Computer Algebra Systems by Schwarz [1989], Schwarz [1994], Bronstein [1994], Bronstein and Petkovšek [1996].

In 1894 Beke gave a method for factorization of linear differential operators in the ring $\overline{\mathbb{Q}}(x)[\partial]$. The original idea of Beke's algorithm is:

"To decide, in finitely many steps, if a linear homogeneous differential equation is reducible or not, and to construct in the first case an irreducible equation, which has all its solutions in common with the given reducible equation".

In modern language we would say:

"To decide in finitely many steps if a differential operator is reducible or not, and to construct in the first case a non-trivial right-hand factor".

To find a factor via Beke's algorithm one must first compute another operator (the second exterior power) and then compute a first order right-hand factor, construct an auxiliary operator whose associated Riccati equations have among their solutions all possible coefficients of the possible factor. From the auxiliary operator one can read off degree bounds of the numerators and denominators of these coefficients. The main disadvantage of Beke's approach is its tremendous complexity, originating from the necessity to solve several Riccati equations.

Throughout this section k is a field of characteristic 0, $(k(x), ')$ is equipped with derivation $' = d/dx$, $L \in k(x)[\partial]$ is

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0$$

and $K \supset k(x)$ stands for a Picard-Vessiot extension for $L(y) = 0$. Let

$$V = \{\, f \in K \mid L(f) = 0 \,\}$$

denote the solution space of L in K and let $G(K/k) \subset GL(V)$ be the differential Galois group of L. By Proposition (3), a first order operator $\partial - v$ is a right-hand factor of L if and only if $v \in K$ is a rational solution of the associated Riccati equation, and also if there is $y_1 \in V$ such that $y_1' = vy_1$. Such an y_1 is an exponential solution of the equation $L(y) = 0$.

Beke's algorithm for first order right-hand factors. Beke [1894]:

Input: A linear differential operator

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0.$$

with rational coefficients $a_i \in k(x)$ for all i.

Output: A first order right-hand factor

$$\partial - b \text{ with } b \in k(x).$$

1. Determine the associated Riccati equation $R(x) = 0$ of the corresponding scalar equation $L(y) = 0$.
2. Find a rational solution $b \in k(x)$ of the equation $R(x) = 0$ with the method specified in Section 2.3 or another method mentioned there.
3. If none exists, return "There is no first order right-hand factors" and end, else return "$\partial - b$ is the right-hand factor" and end.

What happens now if we want to have a right-hand factor of order m with $1 < m < n$?

Suppose that there exist another operator $M \in k(x)[\partial]$ of order $m < n$ with

$$M = \partial^m + b_{m-1}\partial^{m-1} + \cdots + b_1\partial^1 + b_0\partial^0$$

such that M is a right factor of L. In other words, the linear homogeneous differential equations $M(y) = 0$, i.e.,

$$y^{(m)} + b_{m-1}y^{(m-1)} + \cdots + b_1 y' + b_0 y = 0, \tag{3.4.7}$$

of order m, which has all its solutions in common with the scalar equation $L(y) = 0$.

Let $\{y_1, \ldots, y_m\}$ be a fundamental set of solutions of Equation (3.4.7), as we already know by Proposition (3), the coefficients of Equation (3.4.7) can be represented by this fundamental set in terms of the Wronskians in the following way

$$b_{m-r} = -\frac{w_{m-r}}{w} \text{ for all } r = 1, \ldots, m$$

where $w = w(y_1, \ldots, y_m) = w_m$ is the Wronskian of $y_1, \ldots, y_m$ and w_{m-r} is obtained from w by replacing $y_1^{(m-r)}$ by $y_1^{(m)}$, $y_2^{(m-r)}$ by $y_2^{(m)}$ and so on, for all $r = 1, \ldots, m$. In particular we have:

$$b_{m-1} = -\frac{w'}{w}.$$

in other words, w is an exponential solution of an linear differential equation.

Now the question is:

How can we find a set of equations, actually linear homogeneous differential equations, whose coefficients can be described in a rational way by the coefficients of the the given equation?

The easy way to find them is the following:

1. Write the derivative of each generalized Wronskian as a linear combination of the Wronskians themselves. Differentiating the determinants w_i according to the independent variable, we obtain as a result a linear group of the Wronskians. In particular,

$$w'_0, \ldots, w'_m \in k[w_0, \ldots, w'_m]$$

actually, the coefficients of the w_i in the expression of each w_i are polynomials in the coefficients of the original equation.

2. By successive differentiation and suitable elimination, taking into account that the set of Wronskians is closed under differentiation if the original equation is used to substitute derivatives of order higher than $n-1$, for each of the Wronskians we obtain an $\binom{n}{m}$-th order linear differential equation, i.e., there exist $c_{i,0}, \ldots, c_{i,\binom{n}{m}} \in k$ such that

$$\sum_{j=0}^{\binom{n}{m}} c_{i,j} w_i^{(j)} = 0 \text{ for all } i.$$

These equations are called **associated equations**.

Once we have the associated equations, we want to determine b_{m-1}. To this aim we consider the associated equation of $w = w_m$ and search for exponential solutions, i.e., for rational solutions of the associated Riccati equation.

For a rational coefficient b_{m-1} to exist the equation for w_{m-1} must have a solution with a rational logarithmic derivative due to Liouville's relation. If this is true, the equations for the remaining coefficients b_i are obtained from the associated equations for the Wronskians w_i. If each of these equations has a rational solution, a candidate for a right hand factor of the equation $L(y) = 0$ has been found.

First we build this way the differential equation for w

$$c_{m,\sigma}w^{\sigma} + c_{m,\sigma-1}w^{(\sigma-1)} + \cdots + c_{m,0}w = 0 \text{ with } \sigma = \binom{n}{m}. \tag{3.4.8}$$

In order that Equation (3.4.7) exists, $b_{m-1} = \dfrac{w'}{w}$ has to be rational. This means, Equation (3.4.8) *has to have a solution whose logarithmic derivative is rational.* If there is no such solution, we stop the process.

Beke's algorithm for right factors of higher order. Beke [1894]:

Input: A linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1y' + a_0y = 0$$

with $a_i \in k(x)$ for all i and $m < n$.

Output: A right-hand factor

$$y^{(m)} + b_{m-1}y^{(m-1)} + b_{m-2}y^{(m-2)} + \cdots + b_1y' + b_0y = 0$$

with $b_i \in k(x)$. If no genuine factor exists, the input equation is returned unchanged.

1. Determine the associated equations.
2. Determine a solution of the equation for b_{m-1} found in step 1 with rational logarithmic derivative and determine b_{m-1} from it. If none exists then end.
3. Determine the equation for b_j.
4. Find rational solutions of the equation determined in step 3 and determine b_j from it. If none exists, then end, else go to step 5.
5. From the coefficients b_j, construct a right hand factor of the left side of the input equation and return.

Example 7. *Let $L \in k(x)[\partial]$ be the operator*

$$L = \partial^3 + a_2\partial^2 + a_1\partial + a_0\partial^0,$$

and consider the scalar equation $L(y) = 0$, i.e.,

$$y^{(3)} + a_2y^{(2)} + a_1y' + a_0y = 0.$$

Suppose first that there exists a first order operator $M_1 = \partial + b\partial^0$. Let $y_0 \in k(x)$ be a solution of the equation $M_1(y) = 0$. Then, $b = \frac{y_0'}{y_0}$, and $b \in k(x)$ is a rational solution of the associated Riccati equation

$$R_3(u) = P_3\left(u, u', u^{(2)}\right) + a_2P_2\left(u, u', u^{(n-2)}\right) + a_1P_1(u, u', u^{(2)}) + a_0 = 0$$

where the P_i are polynomials with integer coefficients satisfying

$$P_0 = 1, \text{ and } P_i = P_{i-1}' + uP_{i-1} \text{ for all } i \geq 1.$$

Now, suppose that there exists a second order right factor operator

$$M_2 = \partial^2 + b_1\partial + b_0\partial^0.$$

Let $\{y_1, y_2\} \in k(x)$ be a fundamental set of solutions of $M_2(y) = 0$, and consider the generalized Wronskians:

$$\omega_0 = \begin{vmatrix} y_1' & y_2' \\ y_1^{(2)} & y_2^{(2)} \end{vmatrix}, \ \omega_1 = \begin{vmatrix} y_1 & y_2 \\ y_1^{(2)} & y_2^{(2)} \end{vmatrix}, \ \omega_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The associated linear system is

$$\begin{cases} \omega_0' &= -a_2\omega_0 + a_0\omega_2 \\ \omega_1' &= \omega_0 - a_2\omega_1 - a_1\omega_2 \\ \omega_2' &= \omega_1 \end{cases}.$$

Therefore, the associated equations are:

$$\begin{cases} (a_0)^2\omega_0^{(3)} + 2(a_0a_2 - a_0')a_0\omega_0^{(2)} + \\ \left\{a_0(a_2a_0 - a_0')a_2 + 2a_0(a_2'a_0 - a_2a_0') + (a_0)^2a_1 - \left[a_0a_0^{(2)} - 2a_0(a_0')^2\right]\right\}\omega_0' + \\ \left[a_1a_2(a_0)^2 + a_0(a_2'a_0 - a_2a_0')a_2 + (a_0)^2a_2^{(2)} - a_0a_0^{(2)}a_2\right. \\ \left. -2a_0a_0'a_2' - 2(a_0')^2a_2 - (a_0)^3\right]\omega_0 = 0, \\ \\ a_0^2\omega_1^{(3)} + 2a_0(a_0a_2 - a_0')\omega_1^{(2)} + \\ [a_0^2a_2^2 + a_0^2a_1 + 2a_0^2a_2' - 3a_0a_0'a_2 + 2(a_0')^2 - a_0a_0^{(2)}]\omega_1' + \\ [a_0^2a_1a_2 - a_0^3 + a_0^2a_2a_2' - a_0a_0'a_2^2 - 2a_0a_0'a_2' + 2(a_0')^2 - a_0a_0^{(2)}a_2 + a_0^2a_2^{(2)}]\omega_1 = 0 \\ \\ \omega_2^{(3)} + 2a_2\omega_2^{(2)} + (a_1 + a_2^2 + a_2')\omega_2' + (-a_0 + a_1a_2 + a_1')\omega_2 = 0 \end{cases}$$

In order words,

$$\begin{cases} \omega_0^{(3)} + P_{0,2}\omega_0^{(2)} + P_{0,1}\omega_0' + P_{0,0}\omega_0 = 0 \\ \\ \omega_1^{(3)} + P_{1,2}\omega_1^{(2)} + P_{1,1}\omega_1' + P_{1,0}\omega_1 = 0 \\ \\ \omega_2^{(3)} + P_{2,2}\omega_2^{(2)} + P_{2,1}\omega_2' + P_{2,0}\omega_2 = 0 \end{cases},$$

where

$$P_{0,2} = \frac{2(a_0a_2 - a_0')a_0}{(a_0)^2},$$

$$P_{0,1} = \frac{a_0(a_2a_0 - a_0')a_2 + 2a_0(a_2'a_0 - a_2a_0') + (a_0)^2a_1 - \left[a_0a_0^{(2)} - 2a_0(a_0')^2\right]}{(a_0)^2},$$

$$P_{0,0} = \frac{a_1a_2(a_0)^2 + a_0(a_2'a_0 - a_2a_0')a_2 + (a_0)^2a_2^{(2)} - a_0a_0^{(2)}a_2 - 2a_0a_0'a_2' - 2(a_0')^2a_2 - (a_0)^3}{(a_0)^2},$$

$$P_{1,2} = \frac{2a_0(a_0a_2 - a_0')}{a_0^2},$$

$$P_{1,1} = \frac{a_0^2a_2^2 + a_0^2a_1 + 2a_0^2a_2' - 3a_0a_0'a_2 + 2(a_0')^2 - a_0a_0^{(2)}}{a_0^2},$$

$$P_{1,0} = \frac{a_0^2a_1a_2 - a_0^3 + a_0^2a_2a_2' - a_0a_0'a_2^2 - 2a_0a_0'a_2' + 2(a_0')^2 - a_0a_0^{(2)}a_2 + a_0^2a_2^{(2)}}{a_0^2},$$

$P_{2,2} = 2a_2,\ P_{2,1} = (a_1 + a_2^2 + a_2')$, *and* $P_{2,0} = (-a_0 + a_1a_2 + a_1')$.

Finally, the associated Riccati equations are

$$\begin{cases} u_0^3 + 3u_0'u_0 + u_0^{(2)} + P_{0,1}(u_0^2 + u_0') + P_{0,2}u_0 + P_{0,0} = 0 \\ u_1^3 + 3u_1'u_1 + u_1^{(2)} + P_{1,1}(u_1^2 + u_1') + P_{1,2}u_1 + P_{1,0} = 0 \\ u_2^3 + 3u_2'u_2 + u_2^{(2)} + P_{2,1}(u_2^2 + u_2') + P_{2,2}u_2 + P_{0,0} = 0 \end{cases}.$$

3.5 *The Schwarz's* LODEF *Algorithm*

Schwarz made an analysis of the costs of factorizing linear homogeneous differential equations with rational coefficients. He then described the algorithm of Beke differently. He recursively reduces the order of possible right factors, beginning with order n - 1, ending with the search for first order factors. Moreover, he estimates bounds for the degree of the coefficients of possible right factors and computes the size of polynomial and rational solutions of certain differential equations. In this way he developed the RiccatiRational algorithm to complete the last step of the Beke algorithm, that is, solving the generalized Riccati equations derived from the associate equations.

To factor operators over k we proceed as follows:

Let $L = L_2L_1$ where L_2 has order $n-r$ and L_1 has order r. Since the solutions of $L_1(y) = 0$ are also solutions of $L(y) = 0$, we can write

$$L_1(y) = y^{(r)} + b_{r-1}y^{(r-1)} + \cdots + b_0y = \frac{\begin{vmatrix} y & y_1 & \cdots & y_r \\ y' & y_1' & \cdots & y_r' \\ \vdots & \vdots & \cdots & \vdots \\ y^{(r)} & y_1^{(r)} & \cdots & y_r^{(r)} \end{vmatrix}}{\begin{vmatrix} y_1 & \cdots & y_r \\ y_1' & \cdots & y_r' \\ \vdots & \cdots & \vdots \\ y_1^{(r-1)} & \cdots & y_r^{(r-1)} \end{vmatrix}}$$

where $y_1, \dots, y_r$ are solutions of $L(y) = 0$. Note that the denominator of the right-hand side of the latter equation is the Wronskian of a fundamental set of solutions of L_1. Therefore it is exponential over k. Consider $L \in \mathbb{Q}(x)[\partial]$,

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0$$

and let $\{y_1, \dots, y_n\}$ be a fundamental set of solutions of $L(y) = 0$. Build the matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_m^{(n-1)} \end{pmatrix} \tag{3.5.9}$$

where $m < n$. We denote the determinants of the $m \times m$ sub-matrices of (3.5.9) by $z_1, \dots, z_{\binom{n}{m}}$. The set $\{z_\nu \mid \nu = 1, \dots, \binom{n}{m}\}$ may be considered as a set of new functions, which is closed under differentiation if the original differential equation is used to substitute derivatives of order higher than $n-1$.

By suitable differentiations and elimination for each of these functions, an $\binom{n}{m}$-th order linear differential equation may be obtained. These equations are called **associated equations** for the original one.

Procedure. Schwarz [1989]: The full range for the index ν is subdivided into $n-m+1$ subintervals $I^{(k)}$ which are defined by

$$I^{(k)} = \left\{ \nu \,\middle|\, \binom{k}{m} + 1 \le \nu \le \binom{k+1}{m},\ m-1 \le k \le n-1 \right\}$$

An index ν belongs to $I^{(k)}$ if in the corresponding z_ν the highest derivatives of $y_1, y_2, \dots, y_n$ are exactly of order k. Each z'_ν may be expressed as a linear homogeneous function of the z_ν, where $\nu = 1, \dots, \binom{n}{m}$, by using the original differential Equation (3.1.1) for substituting derivatives of order higher than $n-1$. As we enumerate with respect to z, these relations may be written in the form

$$z'_\nu = \sum_{\mu \in I^{(k)} \cup I^{(k+1)}} \alpha_{\nu\mu} z_\mu,$$

where $\alpha_{\nu\mu}$ is integer for $\nu \in I^{(k)}$, $m-1 \le k \le n-2$, and

$$z'_\nu = \sum_{\mu=1}^{\binom{n}{m}} \alpha_{\nu\mu} z_\mu,$$

where $\alpha_{\nu\mu}$ is linear homogeneous in $a_0, \dots, a_{n-1}$ for $\nu \in I^{(n-1)}$. The associated equations are generalizations of Liouville's relation.

Example 8. *For $n = 3$ we have the equation*

$$y^{(3)} + a_2 y^{(2)} + a_1 y' + a_0 y = 0.$$

If $m = 1$ then $k = 1$, and

$$I^{(1)} = \left\{ \nu \,\middle|\, \binom{1}{1} + 1 \le \nu \le \binom{1+1}{1} \right\} = \{2\}.$$

If $m = 2$ then $k = 1$ or 2, and

- *for $k = 1$,*

$$I^{(1)} = \left\{ \nu \,\middle|\, \binom{1}{2} + 1 \le \nu \le \binom{1+1}{2} \right\} = \{1\},$$

- *for $k = 2$,*

$$I^{(2)} = \left\{ \nu \,\middle|\, \binom{2}{2} + 1 \le \nu \le \binom{2+1}{2} \right\} = \{2,\ 3\}.$$

There are three second-order determinants,

$$z_1 = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix},\ z_2 = \begin{vmatrix} y_1 & y_2 \\ y_1^{(2)} & y_2^{(2)} \end{vmatrix},\ z_3 = \begin{vmatrix} y'_1 & y'_2 \\ y_1^{(2)} & y_2^{(2)} \end{vmatrix}.$$

From these expressions we get

$$z'_1 = z_2,\ z'_2 = -a_1 z_1 - a_2 z_2 + z_3,\ w'_3 = -a_0 z_1 - a_2 z_3.$$

Therefore, the third order equations for w_1 and w_1 are

$$z_1^{(3)} + 2a_2 z_1^{(2)} + (a_1 + a_2^2 + a'_2) z'_1 + (-a_0 + a_1 a_2 + a'_1) z_1 = 0 \tag{3.5.10}$$

$$\begin{gathered} a_0^2 z_2^{(3)} + 2a_0(a_0 a_2 - a'_0) z_2^{(2)} + \\ [a_0^2 a_2^2 + a_0^2 a_1 + 2a_0^2 a'_2 - 3a_0 a'_0 a_2 + 2(a'_0)^2 - a_0 a_0^{(2)}] z'_2 + \\ [a_0^2 a_1 a_2 - a_0^3 + a_0^2 a_2 a'_2 - a_0 a'_0 a_2^2 - 2a_0 a'_0 a'_2 + 2(a'_0)^2 - a_0 a_0^{(2)} a_2 + a_0^2 a_2^{(2)}] z_2 = 0. \end{gathered} \tag{3.5.11}$$

Algorithm LODEF Schwarz [1989]:

Input: A linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1y' + a_0y = 0 \tag{3.5.12}$$

with $a_i \in \mathbb{Q}(x)$ for all $i = 0, \ldots, n-1$.

Output: A right-hand factor

$$y^{(m)} + b_{m-1}y^{(m-1)} + b_{m-2}y^{(m-2)} + \cdots + b_1y' + b_0y = 0$$

of order $m < n$ with $b_i \in \mathbb{Q}(x)$. If no genuine factor exists, the input equation is returned unchanged.

$m := 0$, $j := 1$

1. $m := m + 1$. If $m = n$ return the input equation.
2. Determine the associated equations.
3. Determine a solution of the equation for b_{m-1} found in step 2 with rational logarithmic derivative and determine b_{m-1} from it. If none exists, go to step 1.
4. $j := j + 1$. If $j > m$ go to step 7.
5. Determine the equation for b_j.
6. Find rational solutions of the equation determined in step 5 and determine b_j from it. If none exists, go to step 1, else go to step 4.
7. From the coefficients b_j, construct a factor of left side of Equation (3.5.12) and return.

The above yields all the possible candidates for right factors of order m, and trial divisions either determine an actual factor, or prove that there is no such factor.

Example 9. *Let us consider the equation*

$$L_2(y) = y^{(3)} + \frac{x-1}{x}y^{(2)} + \frac{x^2-2}{x}y' + \frac{2}{x^2}y = 0,$$

where

$$a_2 = \frac{x-1}{x}, \ a_1 = \frac{x^2-2}{x}, \ a_0 = \frac{2}{x^2}.$$

Differentiating a_2, a_1 and a_0, we get

$$a_2' = \frac{1}{x^2}, \ a_1' = \frac{x^2+2}{x^2}, \ a_0' = \frac{-4}{x^3}, \ a_2^{(2)} = \frac{-2}{x^3}, \ a_1^{(2)} = \frac{-4}{x^3}, \ a_0^{(2)} = \frac{12}{x^4}.$$

In this case $n = 3$, and we can take $m = 2$. By Equation (3.5.10), the corresponding associated equation for z_1 is:

$$z_1^{(3)} + 2\left(\frac{x-1}{x}\right)z_1^{(2)} + \left(\frac{x^3+x^2-4x+2}{x^2}\right)z_1' + \left(\frac{x^3-2x+2}{x^2}\right)z_1 = 0. \tag{3.5.13}$$

By Equation (3.5.10) the corresponding associated equation for z_2 is:

$$z_2^{(3)} + \frac{2(x+1)}{x}z_2^{(2)} + \frac{(x^3+x^2+2x-1)}{x^2}z_2' + \frac{(x^4-x^3-x+1)}{x^3}z_2 = 0. \tag{3.5.14}$$

The generalized Riccati equation associated to Equation (3.5.13) is:

$$u^{(2)} + \left[3u + 2\left(\frac{x-1}{x}\right)\right]u' + u^3 + 2\left(\frac{x-1}{x}\right)u^2 + \left(\frac{x^3+x^2-4x+2}{x^2}\right)u + \left(\frac{x^3-2x+2}{x^2}\right) = 0,$$

which has solution

$$u = 1 \Leftrightarrow \frac{y'}{y} = 1 \Leftrightarrow \frac{\omega'}{\omega_{S_3}} = 1.$$

By the Liouville's relation, Equation (3.1.4) , we obtain $r_1 = -1$. *To obtain the equation for* r_2, *substitute*

$$z_2 = ve^{r_1} = ve^{-1}$$

in the associated Equation (3.5.14). We get

$$v^{(3)} - \frac{(x-2)}{x}v^{(2)} + \frac{(x^3-x-1)}{x^2}v' - \frac{(x^3-1)}{x^3}v = 0,$$

and its generalized Riccati associated equation is:

$$u^{(2)} + \left[3u - \frac{(x-2)}{x}\right]u' + u^3 - \frac{(x-2)}{x}u^2 + \frac{(x^3-x-1)}{x^2}u - \frac{(x^3-1)}{x^3} = 0.$$

The solution of the generalized Riccati equation is

$$v = \frac{x^2-1}{x}, \text{ and then } r_2 = \frac{x^2-1}{x}.$$

Therefore,

$$y^{(2)} + y' + \frac{(x^2-1)}{x}y = \partial^2(y) + \partial(y) + \frac{(x^2-1)}{x}\partial^0(y) =$$

$$\left[\partial^2 + \partial + \frac{(x^2-1)}{x}\partial^0\right](y)$$

is a right factor of the equation. Now, in order to find a left factor in this case we subtract the derivative of the right factor from the original equation and rearrange the terms to obtain:

$$-\frac{1}{x}\left(y^{(2)} + y' + \frac{(x^2-1)}{x}y\right).$$

Then, $\partial - \frac{1}{x}$ *is the left factor. We can factor* $L_2(y)$, *as:*

$$L_2 = \left(\partial - \frac{1}{x}\partial^0\right)\left[\partial^2 + \partial + \frac{(x^2-1)}{x}\partial^0\right].$$

Remark 3.5.1. *We have solved the last Riccati equation with the algorithm RiccatiRational of Schwarz [1994], explained in section* 3.

3.6 The Bronstein's Algorithm

In this section we present the efficient algorithm due to Bronstein [1994], for computing the associated equations appearing in the Beke factorization method. It produces several possible associated equations, of which only the simplest can be selected for solving.

Let k be a field of characteristic 0, $n, m \in \mathbb{Z}$ with $n \geq m > 0$, $A \in k^{n\times m}$ a matrix with entries in k, given by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}_{n\times m}$$

and any set

$$S = \{ s_1, \dots, s_m \}$$

of m positive integers with

$$1 \leq s_1 < \cdots < s_m \leq n.$$

Let A_S denote the square sub-matrix obtained from the rows $s_1, \ldots, s_m$

$$A_S = \begin{pmatrix} a_{s_1 1} & a_{s_1 2} & \cdots & a_{s_1 m} \\ a_{s_2 1} & a_{s_2 2} & \cdots & a_{s_2 m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s_m 1} & a_{s_m 2} & \cdots & a_{s_m m} \end{pmatrix}_{m \times m}$$

and let ω_{A_S} be its determinant. Let $' : k \to k$ be a derivation and consider $L \in k(x)[\partial]$,

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0.$$

Let $K \supset k(x)$ be a Picard-Vessiot extension for $L(y) = 0$, $V = \{y \in K \mid L(y) = 0\}$ the solution space of L in K and $G \subset GL(V)$ the differential Galois group of L. Suppose that there exists another operator $M \in k(x)[\partial]$ of order $m < n$ with

$$M = \partial^m + b_{m-1}\partial^{m-1} + \cdots + b_1\partial + b_0\partial^0 = 0$$

such that M is a right factor of L. In other words, the scalar equation

$$M(y) = y^{(m)} + b_{m-1}y^{(m-1)} + \cdots + b_1 y + b_0 y^{(0)} = 0$$

of order m has all its solutions in common with the scalar equation $L(y) = 0$. Let $\{y_1, \ldots, y_m\} \subset K$ be the fundamental set of solutions of the scalar equation $M(y) = 0$, and define the n-th generalized Wronskian matrix of $\{y_1, \ldots, y_m\}$ to be the $n \times m$ matrix:

$$W = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_m^{(n-1)} \end{pmatrix}_{n \times m}. \tag{3.6.15}$$

Now, consider the following sets:

$$S_0 = \{\, 2, \ldots, m+1 \,\},$$

$$S_i = \{\, 1, \ldots, i \,\} \cup \{\, i+2, \ldots, m+1 \,\}, \text{ for } i = 0, \ldots, m, \text{ and} \tag{3.6.16}$$

$$S_m = \{\, 1, \ldots, m \,\}.$$

Then

$$\omega_{S_i} = \begin{vmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(i-1)} & y_2^{(i-1)} & \cdots & y_m^{(i-1)} \\ y_1^{(i+1)} & y_2^{(i+1)} & \cdots & y_m^{(i+1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(m-1)} & y_2^{(m-1)} & \cdots & y_m^{(m-1)} \\ y_1^{(m)} & y_2^{(m)} & \cdots & y_m^{(m)} \end{vmatrix}.$$

In particular, W_{S_m} is the usual Wronskian matrix of $\{y_1, \ldots, y_m\}$. In fact, in this case we have $S_m = \{1, \ldots, m\}$ with $n = m$. Then

$$W_{S_m} = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(m-1)} & y_2^{(m-1)} & \cdots & y_m^{(m-1)} \end{pmatrix}_{m \times m}$$

and

$$\omega_{S_m} = \begin{vmatrix} y_1 & y_2 & \dots & y_m \\ y_1' & y_2' & \dots & y_m' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(m-1)} & y_2^{(m-1)} & \dots & y_m^{(m-1)} \end{vmatrix}.$$

Example 10. *If $m = 2$ then for $i = 0, 1, 2$ we have,*

$$S_0 = \{2, 3\}, S_1 = \{1\} \cup \{3\} = \{1, 3\}, \text{ and } S_2 = \{1, 2\}$$

where

$$\omega_{S_0} = \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}, \omega_{S_1} = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}, \text{ and } \omega_{S_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

If $m = 3$ then for $i = 0, 1, 2, 3$ we have,

$$S_0 = \{2, 3, 4\}, S_1 = \{1\} \cup \{3, 4\} = \{1, 3, 4\}, S_2 = \{1, 2\} \cup \{4\} = \{1, 2, 4\},$$

$$\text{and } S_3 = \{1, 2, 3\}$$

where

$$\omega_{S_0} = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}, \omega_{S_1} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix},$$

$$\omega_{S_2} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}, \text{ and } \omega_{S_3} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \end{vmatrix}.$$

From Theorem 5 we know that, if the set $\{y_1, \dots, y_m\}$ is a fundamental system of solutions of $M(y) = 0$ then

$$\omega_{S_m} \neq 0$$

and for each $i = 0, \dots, m-1$ exists then $c_{i,0}, \dots, c_{i,\binom{n}{m}} \in K$ such that

$$\sum_{j=0}^{\binom{n}{m}} c_{i,j} \omega_{s_i}^{(j)} = 0.$$

This means that each ω_S satisfies a linear ordinary differential equation of order at most $\binom{n}{m}$ with coefficients in K. These equations are called the m-th **associated equations** of L.

Let S be a set of positive integers and assume that it is sorted in increasing order. Define the following operations in S:

Increment the i-th element of S: $S_i^+ = (S \cup \{1 + s_i\}) \setminus \{s_i\}$.

Replace the i-th element of S by l: $S_i^{[l]} = (S \cup \{l\}) \setminus \{s_i\}$.

Number of elements of S which are strictly between l and s_i:

$$\delta_i^{[l]}(S) = \#\{s \in S : l < s < s_i\}.$$

We only need to manipulate the subsets of $\{1, \dots, n\}$ for a given integer n (the order of the operator we want to factorize), and we can apply the above set of operations to minors of a rectangular matrix:

Let R be any commutative ring, and A an $n \times m$ matrix with coefficients in R where $n \geq m$ given by:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}_{n\times m}.$$

For any set $S = \{ s_1, \dots, s_m \}$ of m integers with

$$1 \leq s_1 < s_2 < \dots < s_m \leq n$$

we define

$$\omega^{+}_{S,i} = \begin{cases} \omega_{A_{S_i^+}} & if\ 1 + s_i \notin S \cup \{ n + 1 \} \\ 0 & if\ 1 + s_i \in S \cup \{ n + 1 \} \end{cases}. \tag{3.6.17}$$

In particular, if $1 + s_i \notin S \cup \{ n + 1 \}$ then

$$\omega_{A_{S_i^+}} = \begin{vmatrix} a_{s_1 1} & a_{s_1 2} & \dots & a_{s_1 m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s_{i-1} 1} & a_{s_{i-1} 2} & \dots & a_{s_{i-1} m} \\ a_{1+s_i 1} & a_{1+s_i 2} & \dots & a_{1+s_i m} \\ a_{s_{i+1} 1} & a_{s_{i+1} 2} & \dots & a_{s_{i+1} m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s_m 1} & a_{s_m 2} & \dots & a_{s_m m} \end{vmatrix},$$

and for $1 \leq l \leq n$ we define

$$\omega^{[l]}_{S,i} = \begin{cases} (-1)^{\delta_i^{[l]}(S)} \omega_{A_{S_i^{[l]}}} & if\ l \notin S \setminus \{ s_i \} \\ \\ 0 & if\ l \in S \setminus \{ s_i \} \end{cases} \tag{3.6.18}$$

i.e., if $l \notin S \cup \{ s_i \}$ then

$$\omega^{[l]}_{S,i} = (-1)^{\delta_i^{[l]}(S)} \begin{vmatrix} a_{s_1 1} & a_{s_1 2} & \dots & a_{s_1 m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s_{i-1} 1} & a_{s_{i-1} 2} & \dots & a_{s_{i-1} m} \\ a_{l1} & a_{l2} & \dots & a_{lm} \\ a_{s_{i+1} 1} & a_{s_{i+1} 2} & \dots & a_{s_{i+1} m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s_m 1} & a_{s_m 2} & \dots & a_{s_m m} \end{vmatrix}.$$

The following two lemmas, corresponding to Lemma 1 and 2 of Bronstein [1994] respectively, give us explicit formulas for the derivative of each ω_S as a linear combination of the minors. The proofs are straightforward from the properties of the determinants and the previous definitions, for that reason we omit the details.

Lemma 3.

$$n \notin S \ \Rightarrow\ \omega'_S = \sum_{i=1}^{m} \omega^{+}_{S,i}.$$

Lemma 4. *If there exist* $a_1, \dots, a_n \in k$ *such that*

$$y_i^{(n)} = -\sum_{j=1}^{n} a_{n-j} y_i^{(n-j)} \ \textit{for } i = 1, \dots, m, \ \textit{then}$$

$$n \in S \ \Rightarrow\ \omega'_S = \sum_{i=1}^{m-1} \omega^{+}_{S,i} - \sum_{j=0}^{n-1} a_j \omega^{[j+1]}_{S,m}.$$

Let

$$L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0$$

be a linear ordinary differential operator with coefficients in R. For any integer $0 < m < n$ we defined the m-th **associated system** of L to be the first order linear system

$$\omega' = M_m(L)\omega$$

where ω is the vector of all the $m \times m$ minors of W, the generalized Wronskian matrix of $\{y_1, \ldots, y_m\}$, and $L(y_i) = 0$ for $0 \le i \le m$. $M_m(L)$ is the matrix of coefficients of the derivatives of the minors as linear combinations of the minors themselves.

The algorithm for computing the associated system is then:

Bronstein's Algorithm, Bronstein [1994]:

Input: Two integers, one n (the order of the operator we want to factorize), and another m (the order of the possible right factor).

Output: The m-th associated system of a generic operator of order n.

1. Enumerate all the subsets of m integers in $\{1, \ldots, n\}$,

$$\omega = \left(S_1, \ldots, S_{\binom{n}{m}}\right)^T.$$

2. For each i express ω'_{S_i} as

$$(u_{i1}, \ldots, u_{im}) \cdot \omega$$

for some u_{ij}'s in K, using Lemma (3) if $n \notin S_i$, and Lemma (4) if $n \in S_i$.

3. The matrix $M_m(L)$ is then (u_{ij}) for $i, j \in \{1, \ldots, \binom{n}{m}\}$.

Example 11. *Let $L = \partial^4 + a_3\partial^3 + a_2\partial^2 + a_1\partial + a_0\partial^0$ be the generic operator of order 4. Let us compute its second associated system.*

Step 1: Order the subsets of size 2 of $\{1, 2, 3, 4\}$:

$$S_1 = \{1, 2\}, S_2 = \{1, 3\}, S_3 = \{2, 3\}, S_4 = \{1, 4\},$$

$$S_5 = \{2, 4\}, \text{ and } S_6 = \{3, 4\}.$$

Step 2: Apply Lemma 3 to those subsets which do not contain 4:

$$\omega'_{S_1} = \sum_{i=1}^{2} \omega^+_{\{1,2\},i} = \omega^+_{\{1,2\},1} + \omega^+_{\{1,2\},2}.$$

By Equation (3.6.17):

$$\omega^+_{\{1,2\},1} = \begin{cases} \omega_{\{1,2\}^+_1} & if\ 2 \notin \{1, 2\} \cup \{5\} \\ 0 & if\ 2 \in \{1, 2\} \cup \{5\} \end{cases},$$

that is,

$$\omega^+_{\{1,2\},1} = 0$$

and

$$\omega^+_{\{1,2\},2} = \begin{cases} \omega_{\{1,2\}^+_2} & if\ 3 \notin \{1, 2\} \cup \{5\} \\ 0 & if\ 3 \in \{1, 2\} \cup \{5\} \end{cases},$$

where

$$\omega^{+}_{\{1,2\},2} = \omega_{\{1,2\}^{+}_{2}} = \omega_{S_2},$$

since

$$\{\,1,2\,\}^{+}_{2} = (\{\,1,2\,\} \cup \{\,3\,\}) \setminus \{\,2\,\} = \{\,1,3\,\}.$$

Therefore,

$$\omega'_{S_1} = \omega_{S_2} = 0\omega_{S_1} + 1\omega_{S_2} + 0\omega_{S_3} + 0\omega_{S_4} + \omega_{S_5} + 0\omega_{S_6}. \tag{3.6.19}$$

Similarly,

$$\omega'_{S_2} = \omega_{S_3} + \omega_{S_4} = 0\omega_{S_1} + 0\omega_{S_2} + 1\omega_{S_3} + 1\omega_{S_4} + 0\omega_{S_5} + 0\omega_{S_6}, \tag{3.6.20}$$

and

$$\omega'_{S_3} = \omega_{S_5} = 0\omega_{S_1} + 0\omega_{S_2} + 0\omega_{S_3} + 0\omega_{S_4} + 1\omega_{S_5} + 0\omega_{S_6}. \tag{3.6.21}$$

Now apply Lemma (4) to those subsets which do contain 4*:*

$$\omega'_{S_4} = \sum_{i=1}^{1} \omega^{+}_{\{1,4\},i} - \sum_{j=0}^{3} a_j \omega^{[j+1]}_{\{1,4\},2}$$

$$= \omega^{+}_{\{1,4\},1} - a_0\omega^{[1]}_{\{1,4\},2} - a_1\omega^{[2]}_{\{1,4\},2} - a_2\omega^{[3]}_{\{1,4\},2} - a_3\omega^{[4]}_{\{1,4\},2}.$$

By Equation (3.6.17):

$$\omega^{+}_{\{1,4\},1} = \begin{cases} \omega_{\{1,4\}^{+}_{1}} & if\ 2 \notin \{\,1,4\,\} \cup \{\,5\,\} \\ 0 & if\ 2 \in \{\,1,4\,\} \cup \{\,5\,\} \end{cases},$$

where

$$\omega^{+}_{\{1,4\},1} = \omega_{\{1,4\}^{+}_{1}} = \omega_{S_5},$$

since

$$\{\,1,4\,\}^{+}_{1} = (\{\,1,4\,\} \cup \{\,2\,\}) \setminus \{\,1\,\} = \{\,2,4\,\}.$$

Applying Equation (3.6.18) to the other kind of sets we obtain:

$$\omega^{[1]}_{\{1,4\},2} = \begin{cases} (-1)^{\delta^{[1]}_{2}(\{1,4\})}\omega_{\{1,4\}^{[1]}_{2}} & if\ 1 \notin \{\,1,4\,\} \setminus \{\,4\,\} \\ 0 & if\ 1 \in \{\,1,4\,\} \setminus \{\,4\,\} \end{cases},$$

that is,

$$\omega^{[1]}_{\{1,4\},2} = 0;$$

$$\omega^{[2]}_{\{1,4\},2} = \begin{cases} (-1)^{\delta^{[2]}_{2}(\{1,4\})}\omega_{\{1,4\}^{[2]}_{2}} & if\ 2 \notin \{1,4\} \setminus \{4\} \\ 0 & if\ 2 \in \{1,4\} \setminus \{4\} \end{cases}$$

then,

$$\omega^{[2]}_{\{1,4\},2} = (-1)^{\delta^{[2]}_{2}(\{1,4\})}\omega_{\{1,4\}^{[2]}_{2}}$$

where

$$\delta^{[2]}_{2}(\{1,4\}) = \#\,(\{\,s \in \{\,1,4\,\} : 2 < s < 4\,\}) = 0$$

and

$$\{1,4\}^{[2]}_{2} = \{\,1,2,4\,\} \setminus \{\,4\,\} = \{\,1,2\,\}.$$

Therefore

$$\omega^{[2]}_{\{1,4\},2} = \omega_{S_1}.$$

On the other hand,

$$\omega^{[3]}_{\{1,4\},2} = \omega_{S_2} \ and\ \omega^{[4]}_{\{1,4\},2} = \omega_{S_4},$$

whence

$$\omega'_{S_4} = -a_1\omega_{S_1} - a_2\omega_{S_2} + 0\omega_{S_3} - a_3\omega_{S_4} + 1\omega_{S_5} + 0\omega_{S_6}. \tag{3.6.22}$$

Similarly,

$$\omega'_{S_5} = a_0\omega_{S_1} + 0\omega_{S_2} - a_2\omega_{S_3} + 0\omega_{S_4} - a_3\omega_{S_5} + 1\omega_{S_6}, \tag{3.6.23}$$

and

$$\omega'_{S_6} = 0\omega_{S_1} + a_0\omega_{S_2} + a_1\omega_{S_3} + 0\omega_{S_4} + 0\omega_{S_5} - a_3\omega_{S_6}. \tag{3.6.24}$$

Step 3: By Equations (3.6.19), (3.6.20), (3.6.21), (3.6.22), (3.6.23), and (3.6.24) the second associated system for a generic operator of order four is:

$$\omega' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -a_1 & -a_2 & 0 & -a_3 & 1 & 0 \\ a_0 & 0 & -a_2 & 0 & -a_3 & 1 \\ 0 & a_0 & a_1 & 0 & 0 & -a_3 \end{pmatrix} \omega$$

where ω is the column vector

$$\omega = (\omega_{S_1}, \omega_{S_2}, \omega_{S_3}, \omega_{S_4}, \omega_{S_5}, \omega_{S_6})^T.$$

In conclusion,

$$M_2(L) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -a_1 & -a_2 & 0 & -a_3 & 1 & 0 \\ a_0 & 0 & -a_2 & 0 & -a_3 & 1 \\ 0 & a_0 & a_1 & 0 & 0 & -a_3 \end{pmatrix}. \tag{3.6.25}$$

Now we will see how to go from the associated system to the associated equations. We assume from now on that the coefficients of the operator to factor are in some differential field k.

The following sequence of matrices:

$$M_{m,1}(L), \ldots, M_{m,N}(L)$$

where $N = \binom{n}{m}$ is defined by $M_{m,1}(L) = M_m(L)$, and

$$M_{m,i}(L) = M'_{m,i-1}(L) + M_{m,i-1}(L)M_m(L), \tag{3.6.26}$$

for $2 \leq i \leq N$, where $'$ means point-wise differentiation.

Example 12. *For $i = 2$ we have*

$$M_{m,2}(L) = M'_{m,1}(L) + M_{m,1}(L)M_m(L).$$

Substituting $M_{m,1}$ we get

$$M_{m,2}(L) = M'_m(L) + M_m(L)M_m(L), \text{ i.e.,}$$

$$M_{m,2}(L) = M'_{m,1}(L) + [M_m(L)]^2.$$

Example 13. *Consider following the operator*

$$L = \partial^4 - 2x\partial^2 - 2\partial + x^2\partial^0$$

with coefficients

$$a_3 = 0, a_2 = -2x, a_1 = -2, \text{ and } a_0 = x^2.$$

By the matrix (3.6.25), the associated matrix of the second associated system is:

$$M_2(L) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 2x & 0 & 0 & 1 & 0 \\ x^2 & 0 & 2x & 0 & 0 & 1 \\ 0 & x^2 & -2 & 0 & 0 & 0 \end{pmatrix},$$

and by Equations (3.6.26) the corresponding sequence of matrices is:

$$M_{2,1}(L) = M_2(L),$$

$$M_{2,2}(L) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2x & 0 & 0 & 2 & 0 \\ x^2 & 0 & 2x & 0 & 0 & 1 \\ x^2 & 4 & 4x & 2x & 0 & 1 \\ 2x & 2x^2 & 0 & 0 & 2x & 0 \\ 0 & 2x & x^2 & x^2 & -2 & 0 \end{pmatrix},$$

$$M_{2,3} = \begin{pmatrix} 2 & 2x & 0 & 0 & 2 & 0 \\ 2x^2 & 4 & 6x & 2x & 0 & 2 \\ 2x & 2x^2 & 0 & 0 & 2x & 0 \\ 6x & 6x^2 & 6 & 6 & 6x & 0 \\ 2+2x^2 & 6x & 6x^2 & 2x^2 & 2 & 2x \\ 0 & 2+2x^2 & 0 & 4x & 2x^2 & -2 \end{pmatrix},$$

$$M_{2,4} = \begin{pmatrix} 2x^2 & 4 & 6x & 2x & 0 & 2 \\ 8x & 8x^2 & 6 & 6 & 8x & 0 \\ 2+2x^3 & 6x & 6x^2 & 2x^2 & 2 & 2x \\ 18+6x^3 & 30x & 18x^2 & 6x^2 & 18 & 6x \\ 12x^2 & 8+8x^3 & 18x & 10x & 8x^2 & 4 \\ 8x+2x^4 & 12x^2 & 6+6x^3 & 6+2x^3 & 8x & 2x^2 \end{pmatrix},$$

$$M_{2,5} = \begin{pmatrix} 8x & 8x^2 & 6 & 6 & 8x & 0 \\ 20+8x^3 & 36x & 24x^2 & 8x^2 & 20 & 8x \\ 12x^2 & 8+8x^3 & 18x & 10x & 8x^2 & 4 \\ 48x^2 & 48+24x^3 & 90x & 42x & 24x^2 & 24 \\ 44x+8x^4 & 60x^2 & 18+24x^3 & 18+8x^3 & 44x & 8x^2 \\ 20+20x^3 & 44x+8x^4 & 42x^2 & 18x^2 & 20+8x^3 & 12x \end{pmatrix},$$

and

$$M_{2,6} = \begin{pmatrix} 20+8x^3 & 36x & 24x^2 & 8x^2 & 20 & 8x \\ 60x^2 & 56+32x^3 & 108x & 52x & 32x^2 & 28 \\ 44x+8x^4 & 60x^2 & 18+24x^3 & 18+8x^3 & 44x & 8x^2 \\ 180x+24x^4 & 228x^2 & 90+72x^3 & 90+24x^3 & 180x & 24x^2 \\ 80+92x^3 & 200x+32x^4 & 204x^2 & 84x^2 & 80+32x^3 & 60x \\ 116x^2+8x^5 & 64+100x^3 & 144x+24x^4 & 80x+8x^4 & 84x^2 & 32+8x^3 \end{pmatrix}.$$

Proposition 4. *Bronstein [1994]. For each $i \geq 2$ we have*

$$\omega^{(i)} = M_{m,i}(L)\omega.$$

For any subset S of m integers in $\{1,\ldots,n\}$, we write ω_S^* for the column vector

$$\left(\omega_S', \omega_S'', \ldots, \omega_S^{(N)}\right)^T \tag{3.6.27}$$

and n_S for the index of S in the chosen ordering of those subsets. Define the matrix A_S to be the $N \times N$ matrix such that the i-th row of A_S is the n_S-th row of $M_{m,i}$ for each i. Then we have

$$A_S\omega = \omega_S^* \tag{3.6.28}$$

and the following situations:

Case 1: If A_S is non-singular then the n_S-th equation in the system

$$\omega_S = A_S^{-1}\omega_S^* \tag{3.6.29}$$

is the associated equation for ω_S, while the other equations give formulas for all the other ω_T's as linear combinations of $\omega_S', \ldots, \omega_S^{(N)}$.

Case 2: If A_S is singular then let $(u_1,\ldots,u_q)$ be the kernel of the transpose of A_S. Since each $u_i \in K^N$ corresponds to a linear dependence of the rows of A_S, each dot product

$$u_i.\omega_S^* \tag{3.6.30}$$

gives an associated equation for ω_S. If $q = N - \text{rank}(A_S) > 1$, we obtain an overdetermined system of associated equations for ω_S, which is handled as one equation with extra conditions.

Lemma 5. *Bronstein [1994]. Let W as in Equation (3.6.15), and suppose that there exist $b_1,\ldots,b_m \in K$ such that*

$$y_i^{(m)} = -\sum_{j=1}^{m} b_{m-j} y_i^{(m-j)} \text{ for } i = 1,\ldots,m.$$

Then, for any set S of m integers in $\{1,\ldots,n\}$ there exist $c_s \in K$ with

$$\omega_S = c_S\omega_{\{1,\ldots,m\}}.$$

As a consequence, either $\omega_S = 0$ or $\frac{\omega_S'}{\omega_S} \in K$ for each S.

Procedure. Bronstein [1994]: To complete the algorithm, we search for right-factors of L of order $m < n$. If there is such a factor, say M, we consider a fundamental set of solutions $\{y_1,\ldots,y_m\}$ of the equation $M = 0$. In this case

$$\omega_{\{1,\ldots,m\}} \neq 0$$

and we compute the associated system and the matrices $M_{m,i}(L)$ given in Equation (3.6.26). We look for a subset S such that A_S is non-singular.

Case 1: If A_{S_0} is non-singular for some S_0, then we compute the associated equations for ω_{S_0} as described in Equation (3.6.29). We are interested only in solutions whose logarithmic derivatives are in K, otherwise, by Lemma (5), $\omega_{S_0} = 0$.

Since $\omega_{\{1,\ldots,m\}}$ is a linear combination of $\omega_{S_0}', \ldots, \omega_{S_0}^{(N)}$, $\omega_{\{1,\ldots,m\}} = 0$, which implies that L has no right factor of order m. Otherwise, we do not need to compute or solve any other associated equation, as we have expressions available for all the other ω_T's as linear combinations of $\omega_{S_0}', \ldots, \omega_{S_0}^{(N)}$, and the candidates for ω_{S_0} yield all the possible candidate factors.

Case 2: If A_S is singular for every S, then we need associated equations for $\omega_{S_0},\ldots,\omega_{S_m}$, where S_i is given by Equation (3.6.16). We first compute the associated equation for $\omega_{S_m} = \omega_{\{1,\ldots,m\}}$ as described in Equation (3.6.29). As before we search only for solutions with logarithmic derivatives in K.

By Gaussian elimination on A_{S_m}, we get an invertible matrix B and an upper triangular matrix U such that $A_{S_m} = BU$, and we can obtain expressions for some other ω_{S_i}'s as linear combinations of $\omega'_{S_m}, \ldots, \omega^{(N)}_{S_m}$ from the equations

$$U\omega = B^{-1}\omega^*_{S_m}.$$

Then we generate the associated equations for the next required ω_{S_i}, and either look for its solutions with logarithmic derivatives in K, or replace $\omega_{S_{m-i}}$ by

$$(-1)^i \omega_{\{1,\ldots,m\}} b_i$$

and search for all the solutions b_i in K. We repeat this process until candidates for all the ω_{S_i}'s are found. Note that after each step, the decomposition

$$A_{S_i} = BU$$

may yield expressions for the other ω_{S_i}'s.

We can summarize the algorithm in the following steps:

Algorithm 2. The Beke-Bronstein Algorithm, Bronstein [1994]:

Input: A linear differential operator

$$\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0\partial^0$$

with coefficients in the field k.

Output: A right factor of order $m < n$

$$\partial^m + b_{m-1}\partial^{m-1} + \cdots + b_1\partial + b_0\partial^0,$$

with coefficients in k. If no genuine factor exists the input operator is returned unchanged.

Look for right factors of order m $(1 \leq m \leq n-1)$, using the pre-computations provided by the Bronstein's Algorithm, as follows:

1. Build an equation whose solution space is spanned by all the Wronskians of order m.
2. Solve for exponential solutions.
3. Test which solutions are Wronskians, and obtain a right factor.

Example 14. *Let us consider again the operator of Example (13)*

$$L = \partial^4 - 2t\partial^2 - 2\partial + t^2\partial^0.$$

Order the 2-element subsets of of $\{1,2,3,4\}$ *as in Example (11), that is,*

$$S_1 = \{1,2\}, S_2 = \{1,3\}, S_3 = \{2,3\}, S_4 = \{1,4\},$$
$$S_5 = \{2,4\}, \text{ and } S_6 = \{3,4\}.$$

According to Equation (3.6.27) we have the following column vectors:

$$\omega^*_{S_1} = \left(\omega'_{S_1}, \omega''_{S_1}, \omega'''_{S_1}, \omega^{(4)}_{S_1}, \omega^{(5)}_{S_1}, \omega^{(6)}_{S_1}\right)^T,$$

$$\omega^*_{S_2} = \left(\omega'_{S_2}, \omega''_{S_2}, \omega'''_{S_2}, \omega^{(4)}_{S_2}, \omega^{(5)}_{S_2}, \omega^{(6)}_{S_2}\right)^T,$$

$$\omega^*_{S_3} = \left(\omega'_{S_3}, \omega''_{S_3}, \omega'''_{S_3}, \omega^{(4)}_{S_3}, \omega^{(5)}_{S_3}, \omega^{(6)}_{S_3}\right)^T,$$

$$\omega^*_{S_4} = \left(\omega'_{S_4}, \omega''_{S_4}, \omega'''_{S_4}, \omega^{(4)}_{S_4}, \omega^{(5)}_{S_4}, \omega^{(6)}_{S_4}\right)^T,$$
$$\omega^*_{S_5} = \left(\omega'_{S_5}, \omega''_{S_5}, \omega'''_{S_5}, \omega^{(4)}_{S_5}, \omega^{(5)}_{S_5}, \omega^{(6)}_{S_5}\right)^T,$$
$$\omega^*_{S_6} = \left(\omega'_{S_6}, \omega''_{S_6}, \omega'''_{S_6}, \omega^{(4)}_{S_6}, \omega^{(5)}_{S_6}, \omega^{(6)}_{S_6}\right)^T. \tag{3.6.31}$$

By the sequence of matrices $M_{2,i}$ for $i = 1, \ldots, 6$ of Example (12),

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2x & 0 & 0 & 2 & 0 \\ 2x^2 & 4 & 6x & 2x & 0 & 2 \\ 8x & 8x^2 & 6 & 6 & 8x & 0 \\ 20+8x^3 & 36x & 24x^2 & 8x^2 & 20 & 8x \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2x & 0 & 0 & 2 & 0 \\ 2x^2 & 4 & 6x & 2x & 0 & 2 \\ 8x & 8x^2 & 6 & 6 & 8x & 0 \\ 20+8x^3 & 36x & 24x^2 & 8x^2 & 20 & 8x \\ 60x^2 & 56+32x^3 & 108x & 52x & 32x^2 & 28 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ x^2 & 0 & 2x & 0 & 0 & 1 \\ 2x & 2x^2 & 0 & 0 & 2x & 0 \\ 2+2x^3 & 6x & 6x^2 & 2x^2 & 2 & 2x \\ 12x^2 & 8+8x^3 & 18x & 10x & 8x^2 & 4 \\ 44x+8x^4 & 60x^2 & 18+24x^3 & 18+8x^3 & 44x & 8x^2 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2 & 2x & 0 & 0 & 1 & 0 \\ x^2 & 4 & 4x & 2x & 0 & 1 \\ 6x & 6x^2 & 6 & 6 & 6x & 0 \\ 18+6x^3 & 30x & 18x^2 & 6x^2 & 18 & 6x \\ 48x^2 & 48+24x^3 & 90x & 42x & 24x^2 & 24 \\ 180x+24x^4 & 228x^2 & 90+72x^3 & 90+24x^3 & 180x & 24x^2 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} x^2 & 0 & 2x & 0 & 0 & 1 \\ 2x & 2x^2 & 0 & 0 & 2x & 0 \\ 2+2x^3 & 6x & 6x^2 & 2x^2 & 2 & 2x \\ 12x^2 & 8+8x^3 & 18x & 10x & 8x^2 & 4 \\ 44x+8x^4 & 60x^2 & 18+24x^3 & 18+8x^3 & 44x & 8x^2 \\ 80+92x^3 & 200x+32x^4 & 204x^2 & 84x^2 & 80+32x^3 & 60x \end{pmatrix},$$

and

$$A_6 = \begin{pmatrix} 0 & x^2 & -2 & 0 & 0 & 0 \\ 0 & 2x & x^2 & x^2 & -2 & 0 \\ 0 & 2+2x^3 & 0 & 4x & 2x^2 & -2 \\ 8x+2x^4 & 12\,t^2 & 6+6x^3 & 6+2x^3 & 8x & 2x^2 \\ 20+20x^3 & 44x+8x^4 & 42x^2 & 18x^2 & 20+8x^3 & 12x \\ 116x^2+8x^5 & 64+100x^3 & 144x+24x^4 & 80x+8x^4 & 84x^2 & 32+8x^3 \end{pmatrix},$$

where A_6 is the only invertible matrix. Its inverse is

$$A_6^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{x}{3} & \frac{x^2(8x^3-31)2}{30(x^3-2)} & \frac{(31x^6-56x^3+48)}{60(x^6-10x^3+16)} & \frac{-x(2x^6-6x^3+19)}{30(x^6-10x^3+16)} \\ 0 & 0 & \frac{1}{6} & \frac{-x(2x^3-5)}{10(x^3-2)} & \frac{-x^2(11x^3-19)}{30(x^6-10x^3+16)} & \frac{(3x^6-8x^3+10)}{60(x^6-10x^3+16)} \\ -\frac{1}{2} & 0 & \frac{x^2}{12} & \frac{-x^3(2x^3-5)}{20(x^3-2)} & \frac{-x^4(11x^3-19)}{60(x^6-10x^3+16)} & \frac{x^2(3x^6-8x^3+10)}{120(x^6-10x^3+16)} \\ \frac{1}{2} & 0 & -\frac{x^2}{12} & \frac{(6x^6+x^3-20)}{60(x^3-2)} & \frac{x(11x^6+7x^3-64)}{60(x^6-10x^3+16)} & \frac{-x^2(x^6-6)}{40(x^6-10x^3+16)} \\ 0 & -\frac{1}{2} & \frac{x}{6} & \frac{-x^2(x^3-5)}{15(x^3-2)} & \frac{-x^3(3x^3-2)}{20(x^6-10x^3+16)} & \frac{x(x^6-x^3+10)}{60(x^6-10x^3+16)} \\ t & -\frac{x^2}{2} & \frac{x^3}{6} - \frac{1}{3} & \frac{-x(2x^6-20x^3+5)}{30(x^3-2)} & \frac{-3x^2(x^6-4x^3+10)}{20(x^6-10x^3+16)} & \frac{(x^9-6x^6+30x^3+10)}{60(x^6-10x^3+16)} \end{pmatrix}.$$

By Equations (3.5.13), (3.6.31) and (3.6.29)

$$\omega_{S_6} = A_{S_6}^{-1}\omega_{S_6}^{*}.$$

By Equation (3.6.29), from the sixth row of A_6^{-1} *we obtain the following associated equation for* ω_{S_6}*:*

$$\omega_{S_6} = x\omega_{S_6}' - \frac{1}{2}x^2\omega_{S_6}'' + \frac{1}{6}(x^3-2)\omega_{S_6}''' - \frac{x(2x^6-20x^3+5)}{30(x^3-2)}\omega_{S_6}^{(4)}$$
$$-\frac{3x^2(x^6-4x^3+10)}{2p(x)}\omega_{S_6}^{(5)} + \frac{x^9-6x^6+30x^3+10}{6p(x)}\omega_{S_6}^{(6)},$$

where $p(x) = 10(x^6-10x^3+16)$*. In particular,*

$$\frac{x^9-6x^6+30x^3+10}{6p(x)}\omega_{S_6}^{(6)} = \frac{3x^2(x^6-4x^3+10)}{2p(x)}\omega_{S_6}^{(5)} + \frac{x(2x^6-20x^3+5)}{30(x^3-2)}\omega_{S_6}^{(4)} \tag{3.6.32}$$

$$-\frac{1}{6}(x^3-2)\omega_{S_6}''' + \frac{1}{2}x^2\omega_{S_6}'' - x\omega_{S_6}' + \omega_{S_6}.$$

For the singular matrices, by Equation (3.6.32), the associated equations for ω_{S_1}*,* ω_{S_2}*, and* ω_{S_3} *are given by the the first, second and third rows of* $A_{S_6}^{-1}$*, according to the chosen ordering of the sets, which yield formulas for the other minors in terms of* ω_{S_6} *and its derivatives:*

$$\omega_{S_1} = -\frac{x(2x^6-6x^3+19)}{3p(x)}\omega_{S_6}^{(6)} + \frac{(31x^6-56x^3+48)}{6p(x)}\omega_{S_6}^{(5)} + \frac{x^2(8x^3-31)}{30(x^3-2)}\omega_{S_6}^{(4)} - \frac{1}{3}x\omega_{S_6}''' + \frac{1}{2}\omega_{S_6}'',$$

$$\omega_{S_2} = \frac{(3x^6-8x^3+10)}{6p(x)}\omega_{S_6}^{(6)} - \frac{x^2(11x^3-19)}{3p(x)}\omega_{S_6}^{(5)} - \frac{x(2x^3-5)}{10(x^3-2)}\omega_{S6}^{(4)} + \frac{1}{6}\omega_{S_6}''',$$

$$\omega_{S_3} = \frac{x^2(3x^6-8x^3+10)}{12p(x)}\omega_{S_6}^{(6)} - \frac{x^4(11x^3-19)}{6p(x)}\omega_{S_6}^{(5)} - \frac{x^3(2x^3-5)}{20(x^3-2)}\omega_{S_6}^{(4)} + \frac{1}{12}x^2\omega_{S_6}''' - \frac{1}{2}\omega_{S_6}'.$$

As the space of solutions of Equation (3.6.30) is generated by the functions x^3-2, x^2 *and* x*, the general solution is given by*

$$\omega_{S_6} = ax^3 + bx^2 + cx - 2a,$$

where a*,* b*, and* c *are arbitrary constants. Hence*

$$\omega_{S_6}' = 3ax^2 + 2bx + c, \omega_{S_6}'' = 6ax + 2b, \omega_{S_6}''' = 6a,$$

$$\text{and } \omega_{S_6}^{(4)} = \omega_{S_6}^{(5)} = \omega_{S_6}^{(6)} = 0.$$

Therefore,

$$\omega_{S_1} = ax + b, \omega_{S_2} = a, \text{ and } \omega_{S_3} = -ax^2 - bx - \frac{c}{2}.$$

Now, if $\partial^2 + b_1\partial + b_2$ *is a possible right factor of* L*, then by Equation (3.1.3) for our particular enumeration of the sets* S *we get*

$$b_1 = -\frac{\omega_{S_2}}{\omega_{S_1}} = -\frac{a}{ax+b}, \text{ and } b_2 = \frac{\omega_{S_3}}{\omega_{S_1}} = \frac{-ax^2-bx-c/2}{ax+b}.$$

Compare with Example (10) for $m = 2$ *in the reverse ordering from 3 to 1. Finally, dividing the operator* L *by*

$$\partial^2 - \frac{a}{ax+b}\partial + \frac{-ax^2-bx-c/2}{ax+b}$$

on the right we obtain the following quotient q *and the remainder* r*:*

$$q = \partial^2 + \frac{a}{(ax+b)}\partial + \left[\frac{-2a^2x^3 - 4abx^2 - 2b^2x - 2a^2 + acx + bc}{2(ax+b)^2}\right],$$

$$\text{and } r = \frac{(4ab + c^2)}{4(ax + b)^2}\partial.$$

For exact right division we get the conditions:

$$4ab + c^2 = 0 \text{ and either } a \neq 0 \text{ or } b \neq 0.$$

Let us take $a = 0$*. Now, in order to have a factorization of the form*

$$L = \partial^4 - 2x\partial^2 + x^2 = L_l L_r,$$

with

$$L_l = \partial^2 - \frac{c^2}{(c^2x - 4b^2)}\partial - \left[\frac{c^4x^3 - 8b^2c^2x^2 + 2b(8x^3 + c^3)x + c^4 - 8b^3c}{(c^2x - 4b^2)^2}\right],$$

and

$$L_r = \partial^2 - \frac{c^2}{(c^2x - 4b^2)}\partial - \left(\frac{c^2x^2 - 4b^2x - 2bc}{c^2x - 4b^2}\right),$$

where b *and* c *are constants not both* 0*, we realize that for* $b = 1$ *and* $c = 0$

$$L_l = \partial^2 - x \text{ and } L_r = \partial^2 - x$$

satisfy the requirements. In conclusion,

$$\partial^4 - 2x\partial^2 - 2\partial + x^2 = (\partial^2 - x)(\partial^2 - x).$$

4. ADVANCED METHODS

In this chapter we will present methods for factoring linear differential operators which are not based on Beke's algorithm. We discuss Singer's eigenring factorization method, introduced in Singer [1996], Newton polygons introduced in Malgrange [1979], and van Hoeij's factorization methods introduced in van Hoeij [1997a] and in van Hoeij [1997b].

Let $\mathbb{K}_p$ denote the field with p elements. The core of Berlekamp's algorithm for factoring a squarefree polynomial $f \in \mathbb{K}_p[x]$ is the structure of the quotient

$$A = \mathbb{K}_p[x]/\mathbb{K}_p[x] \cdot f.$$

In particular, if $f = f_1 \cdots f_m$ where the f_i are pairwise relatively prime irreducible polynomials of degree d_i, then A is the direct sum of fields,

$$A = \mathbb{K}_{p^{d_1}} \oplus \cdots \oplus \mathbb{K}_{p^{d_m}}.$$

If Φ is the map $\Phi : x \mapsto x^p - x$, then $\dim_{\mathbb{K}_p}(\ker \Phi) = m$. Therefore, computing the kernel of the map Φ gives a quick way of determining the number of factors of f and, in particular, irreducibility.

Singer tried to generalize this idea to non-commutative polynomial rings and faced various problems. For example, let k be a field and σ a non-trivial automorphism of k and consider the ring $k[x;\sigma]$ of polynomials in x over k with the usual addition and multiplication defined by

$$x \cdot a = \sigma(a) \cdot x \text{ for all } a \in k.$$

Let $f \in k[x;\sigma]$ and consider the left ideal $k[x;\sigma] \cdot f$. The quotient

$$M = k[x;\sigma]/k[x;\sigma] \cdot f$$

is a left $k[x;\sigma]$-module without a cononical ring structure. Singer replaces M with the ring $\mathcal{E}(M)$ of $k[x;\sigma]$-endomorphisms of M, called the **eigenring** of $k[x;\sigma] \cdot f$. In Griesbrecht [1992] it is shown that f is irreducible if and only if $\mathcal{E}(M)$ has no zero divisors (and in fact is a field).

Considering the ring $\mathcal{D} = k[\partial]$ one can start to proceed as with the ring $k[x;\sigma]$. In contrast to $k[x;\sigma]$, $\mathcal{D}$ has in general a poor supply of two sided ideals. Furthermore, one cannot completely rely on these rings to determine irreducibility. Singer therefore looked beyond purely ring theoretic properties to find criteria for irreducibility. The key fact is that to each linear operator $L \in \mathcal{D}$ one can associate a linear algebraic group $Gal(K/k)$, the Galois group of $L(y) = 0$, where K is its Picard-Vessiot extension, and that L's factorization properties are intimately related with the structure and representation theory of $Gal(K/k)$.

There are distinguished two cases:

1. $Gal(K/k)$ is a reductive group, and
2. $Gal(K/k)$ is a non-reductive group.

When $Gal(K/k)$ is a reductive group, properties of $\text{End}_{\mathcal{D}}(\mathcal{D}/\mathcal{D} \cdot L)$ determine if L is reducible.

Recall that a **linear algebraic group** is a subgroup or the group of invertible $n \times n$ matrices (under matrix multiplication) that is defined by polynomial equations. An **algebraic group** is a group that caries the structure of an algebraic variety, such that the multiplication and inverse are given by regular functions.

A **reductive group** is an algebraic group G such that the unipotent radical of the radical of G is trivial and $G \neq \{e\}$ is connected. The **unipotent radical** of G is the normal subgroup of all unipotent elements in the radical of G.

The **radical** of G is the identity component of the unique maximal normal solvable subgroup of G, which is automatically closed in the Zariski topology. The **identity component** of G, as a topological group, denoted by G° is the unique connected component that contains the identity element e of G. An algebraic group G is **connected** when $G = G^\circ$.

Now for an introduction of the other methods, assume that $f \in \mathbb{C}[[x, y]]$, the ring of formal power series in the indeterminates x and y with complex number coefficients, and suppose that we are interested in solving for y the equation $f(x, y) = 0$, to find a sort of series in $x, y(x)$ such that $f(x, y(x)) = 0$. Suppose that

$$f = \sum_{j,i>0} A_{j,i} x^j y^i$$

is an element of $\mathbb{C}[[x, y]]$, and plot on $\mathbb{R}^2$ the discrete set of points with non-negative integral coordinates

$$\Delta(f) = \{(j, i) \mid A_{j,i} \neq 0\}$$

called the **Newton diagram** of f. The idea is to have a polygonal line whose vertices are points of $\Delta(f)$ and whose sides leave the origin of coordinates and the whole of $\Delta(f)$ in different half-planes, so that we obtain the set

$$\Delta'(f) = \Delta(f) + (\mathbb{R}^+)^2.$$

Then consider the convex hull $\overline{\Delta}(f)$ of $\Delta'(f)$ (i.e., the minimal convex set containing $\Delta'(f)$); the border of $\overline{\Delta}(f)$ consists of two half-lines parallel to the axis and a polygonal line (maybe reduced to a single vertex) joining them. The **Newton polygon** of f, denoted by $N(f)$, is this polygonal line (which has their sides oriented from left to right and from top to bottom).

If the vertices of a Newton polygon, taken according to orientation, are

$$P_l = (m_l, n_l), \ l = 1, \ldots, r,$$

then

$$m_l < m_{l+1} \text{ and } n_l > n_{l+1}, \ l = 1, \ldots, r-1.$$

The **slopes** of the Newton polygon are

$$\mu_l = \frac{n_{l+i} - n_l}{m_{l+1} - m_l} \text{ for } l = 1, \ldots, r-1.$$

The **height** $h(N)$ and the **width** $w(N)$ of the Newton polygon N are defined, respectively, as the maximal ordinate and the maximal abscissa of its vertices, that is,

$$h(N) = n_1 \text{ and } w(N) = m_1.$$

In particular, height and width of the Newton polygons are additive functions. If $n > 1$ is an integer, consider $\mathbb{C}\left((x^{1/n})\right)$ as an extension of $\mathbb{C}((x))$, consistining of elements of the form

$$s = \sum_{j \geq d} a_j x^{j/n}.$$

Let $\mathbb{C}\langle\langle x \rangle\rangle$ stand for the union of all $\mathbb{C}\left((x^{1/n})\right)$. In the sequel $\mathbb{C}\langle\langle x \rangle\rangle$ will be taken as the set of all formal Laurent series

$$s = \sum_{j>>d} a_j x^{j/n} \text{ with } d, n \in \mathbb{Z}, \text{ and } n \geq 1.$$

Define the order of s to be $\text{ord}_x(s) = \infty$ if $s = 0$ and

$$\text{ord}_x(s) = \frac{\min\{\, j \mid a_j \neq 0 \,\}}{n}$$

otherwise. The fractionary power series s with $\text{ord}_x(s) > 0$ are called **Puiseux series**. The main properties of Puiseux series are:

- A Puiseux series $s \in \mathbb{C}\left[\left[x^{1/n}\right]\right]$ is a y-root of $f \in \mathbb{C}\left[[x,y]\right]$ if and only if $y-s$ divides f in $\mathbb{C}\left[\left[x^{1/n},y\right]\right]$.
- Assume that the Newton polygon $N(f)$ has a positive height, otherwise it has no y-roots, with the Newton-Puiseux algorithm, which is an inductive procedure, we can find all y-roots of f.
- The series given rise up to by the Newton-Puiseux algorithm are Puiseux series.
- (**Puiseux's theorem**) If $f \in \mathbb{C}[[x,y]]$ and $h\left(N(f)\right) > 0$, then there is a Puiseux series s which is a y-root of f, namely $f\left(x, s(x)\right)$.

As a consequence the ring $\mathbb{C}\left[[x,y]\right]$ is a unique factorization domain, and the field $\mathbb{C}\left\langle\langle x\rangle\right\rangle$ is algebraically closed.

For further details about the Newton polygons and the Newton-Puiseux algorithm we recommend the book of Casas-Alvero [2000].

A generalization of Newton polygons to the ring $k((x))[\partial]$ is given in Malgrange [1979]. He shows that in the following two cases a differential operator $L \in k((x))[\partial]$ is reducible in this ring and how a factorization can be computed:

1. An operator with broken Newton polygon (i.e., more than one slope).
2. An operator with one slope > 0 where the Newton polynomial is reducible and not a power of an irreducible polynomial.

More recently, van Hoeij [1997a] unified these two cases of factorization and the factorization of **regular singular operators** (operators with only one slope $\mu = 0$ in the Newton polygon), in the so called coprime index 1 factorizations.

Since the elements of $k((x))$ consist of infinitely many terms only a finite number of them can be computed. van Hoeij uses local factorization, whose main ingredients are Newton polygons and Newton polynomials, to factor $L \in k((x))[\partial]$ into $L = QR$ with some accuracy. Coprime index 1 factorization means that $\gcd(Q,R) = 1$ and then the factorization can be lifted by the usual Hensel lifting algorithm.

4.1 The Singer's Factorization Method

In this section we will present an exposition about the first method for factoring linear differential operators not based in the Beke's algorithm, the Singer's eigenring factorization method, which was introduced by Singer [1996].

The eigenring $\mathcal{E}_{\mathcal{D}}(L)$ of a differential equation $L(y) = 0$, is the finite dimensional $\mathcal{C}$-algebra of all endomorphisms of the equation, where $\mathcal{C}$ is the subfield of constants of k. It is the set of all rational solutions of another differential equation associated to L. Singer gives the following method for its computiation:

Suppose the dimension of $\mathcal{E}_{\mathcal{D}}(L)$ is more than 1, then take an element $R \in k[\partial]$ in $\mathcal{E}_{\mathcal{D}}(L)$ which is not a constant (we should take R of order less than $\mathrm{ord}(L)$). Now R is a k-linear map from $V(L)$, the space of solutions of the equation, to $V(L)$. We can obtain a basis of $V(L)$ by computing the matrix of the map R in this basis and take an eigenvalue $c \in k$. Then

$$\mathrm{gcrd}(R-c, L) \in k[\partial]$$

is a non-trivial factor of L.

Let $(k, {}')$ be a differential field of characteristic 0 with algebraically closed field of constants $\mathcal{C}$. Let $\mathcal{D} = k[\partial]$ be the ring of linear ordinary differential operators, that is, the non-commutative polynomial ring in the variable ∂, where

$$\partial a = a\partial + a' \text{ for all } a \in k.$$

For any $L \in \mathcal{D}$ given by

$$L = a_n\partial^n + \cdots + a_0\partial^0 \text{ with } a_n \neq 0,$$

the order of L, denoted by $\mathrm{ord}(L)$ is said to be the integer n and $\mathrm{ord}(0) = -\infty$. The ring $\mathcal{D}$ is both a left and right Euclidean ring, that is, for any $L_1 \neq 0, L_2 \in \mathcal{D}$ there exist unique $Q_r, R_r, Q_l, R_l \in \mathcal{D}$ with

$$\mathrm{ord}(R_r),\ \mathrm{ord}(R_l) < \mathrm{ord}(L_1)$$

such that

$$L_2 = Q_rL_1 + R_r \text{ and } L_2 = L_1Q_l + R_l.$$

For $k \subset K$, the space of solutions of $L(y) = 0$ in K is denote by $\mathrm{Soln}_K(L)$. If

$$\dim_{\mathcal{C}} \mathrm{Sol}_K(L) = \mathrm{ord}(L),$$

then we will say that K contains a full set of solutions of L.

Recall that an element $L \in \mathcal{D}$ of positive order is said to be **reducible** if $L = L_1L_2$ for some operators $L_1, L_2 \in \mathcal{D}$ of positive order. In this case, L_1, L_2 are called factors of L. If L is not reducible, we say that it is **irreducible**.

Given two operators $L_1, L_2 \in \mathcal{D}$ the **greatest common right divisor** of L_1 and L_2, denoted by $\mathrm{gcrd}(L_1, L_2)$ is defined to be the monic non-zero operator of greatest order which divide both operators L_1 and L_2 on the right.

Two operators $L_1, L_2 \in \mathcal{D}$ are said to be **relatively prime** if there is no operator of positive order dividing both on the right.

Given two operators $L_1, L_2 \in \mathcal{D}$ the **least common left multiple** of L_1 and L_2, denoted by $[\,L_1, L_2\,]_l$ is defined to be the monic non-zero operator of smallest order such that both L_1 and L_2 divide this operator on the right. One can extend this definition to the least common left multiple $[\,L_1, \ldots, L_m\,]_l$ of any finite set of operators $\{\,L_1, \ldots, L_m\,\}$.

An operator $L \in \mathcal{D}$ is said to be **completely reducible** if it is a k-left multiple of the least common left multiple of a set of irreducible operators.

The module $\mathcal{D}/\mathcal{D} \cdot L$ is not a ring and one cannot apply Berlekamp techniques directly to this module. A substitute for this module is the ring $\mathrm{End}_{\mathcal{D}}(\mathcal{D}/\mathcal{D} \cdot L)$.

Let $L_1, L_2 \in \mathcal{D}$ and denote by $\overline{R}$ the equivalence class of R in $\mathcal{D}/\mathcal{D} \cdot L_2$, and define

$$\mathcal{E}(L_1, L_2) = \{\,\overline{R} \in \mathcal{D}/\mathcal{D} \cdot L_2 \mid L_1R \text{ is divisible on the right by } L_2\,\}.$$

It is easy to show that this condition depends only on the equivalence class and not on the choice representative. Note that $\mathcal{E}_{\mathcal{D}}(L_1, L_2)$ is closed under addition and multiplication by elements in $\mathcal{C}$.

If $L_1 = L_2 = L$, one can define a multiplication on this vector space and the resulting ring is called the (left) **eigenring** of L and is denoted by $\mathcal{E}_{\mathcal{D}}(L)$, i.e.

$$\mathcal{E}_{\mathcal{D}}(L) = \{\,\overline{R} \in \mathcal{D}/\mathcal{D}.L \mid LR = SL, \text{ for some } S \in \mathcal{D}\,\}$$

The multiplication on $\mathcal{E}_{\mathcal{D}}(L)$ is defined in the following way: for $\overline{R}_1, \overline{R}_2 \in \mathcal{E}_{\mathcal{D}}(L)$, let

$$\overline{R}_1 \cdot \overline{R}_2 = \overline{R_1R_2}.$$

This shows that $\mathcal{E}_{\mathcal{D}}(L)$ is a $\mathcal{C}$-algebra. More important, if L is a completely reducible operator, then:

- L is irreducible in $k\,[\partial]$ if and only if $\mathcal{E}_{\mathcal{D}}(L) = \mathcal{C}$.

- If $\mathcal{E}_{\mathcal{D}}(L)$ contains a non-trivial element $\overline{R}$, then $\mathrm{gcrd}(R-c, L)$ must be a non-trivial right factor of L for some $c \in \mathcal{C}$.

Factoring completely reducible operators thus reduces to computing $\mathcal{E}_{\mathcal{D}}(L)$, which is done in the following way:

Let $n = \mathrm{ord}(L)$, A be the $n \times n$ companion matrix corresponding to L,

$$B = I_n \otimes A - A^T \otimes I_n,$$

where I_n is the $n \times n$ identity matrix, and let $Y \in k^{n^2}$ be the rational solution of $Y' = BY$. If,

$$R = y_0 + y_n\partial + y_{2n}\partial^2 + \cdots + y_{(n-2)n}\partial^{n-2} + y_{(n-1)n}\partial^{n-1}$$

then $\overline{R} \in \mathcal{E}_{\mathcal{D}}(L)$, where $Y = (\, y_0, y_1, \ldots, y_{n^2-n}\,)$.

If one has found an element $\overline{R} \in \mathcal{E}_{\mathcal{D}}(L)$, R of order greater than or equal to 1, then one can produce a non-trivial factor of L. To do this, let $\overline{R} \in \mathcal{E}_{\mathcal{D}}(L)$ be of $\mathrm{ord}(R) \geq 1$. Then LR is divisible on the right by L. Therefore, if z is a solution of $L(y) = 0$, we have that $R(z)$ is again a solution of $L(y) = 0$. This implies that $z \mapsto R(z)$ is a linear map of the solution space of $L(y) = 0$ into itself. If c is an eigenvalue of this map then

$$(R-c)(y) = 0 \text{ and } L(y) = 0$$

have a common solution. Since $0 < \mathrm{ord}(R-c) < n$, the $\mathrm{gcrd}(R-c, L)$ will be a non-trivial factor of L. Therefore, given $\overline{R} \in \mathcal{E}_{\mathcal{D}}(L)$, the condition

$$\mathrm{gcrd}(R-c, L) \neq 1$$

defines a non-empty set of at most n constants and for each of these

$$\mathrm{gcrd}(R-c, L)$$

will be a non-trivial factor of L. The advantage of this approach is that the system $Y' = BY$ is easy to compute and that it is a first order system.

Example 15. *Consider the operator $L = \partial^2$ in $k = \mathbb{C}(x)$. Its corresponding companion matrix is*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix $B = I_2 \otimes A - A^T \otimes I_2$, where I_2 is the 2×2 identity matrix, is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We try to find a rational solution $Y \in \mathbb{C}(x)^{2^2}$ with $Y = (\, y_0, y_1, y_2, y_3\,)$ of the system $Y' = BY$, i.e.,

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \Rightarrow \begin{cases} y_0' = y_1 \\ y_2' = -y_0 + y_3 \\ y_3' = -y_1 \end{cases} \Rightarrow$$

$$\begin{cases} y_0 = \int y_1 dx + c_3 \\ y_3 = -\int y_1 dx + c_2 \\ y_2 = \int \left(-\int -y_1 dx - c_3 - \int y_1 + c_2\right) dx + c_1 \end{cases}.$$

If $y_1 = 0$ and $c_3 = -1$ then by the first equation $y_0 = -1$. On the other hand, if $c_2 = 0$ in the second equation then $y_3 = 0$ and $y_2 = x + c_3$, taking $c_3 = 0$ we get $y_2 = x$. Whence $Y = (-1, 0, x, 0)$. If

$$R = y_0 + y_2\partial = -1 + x\partial$$

then $\overline{R} \in \mathcal{E}(\partial^2)$ with ord$(R) \geq 1$. In other words, L has a right-hand factor. To find it, the condition

$$\textit{gcrd}(x\partial - 1 - c, \partial^2) \neq 1$$

gives us the non-empty set $\{ c_1 = 0, c_2 = -2 \}$ such that

$$\textit{gcrd}(R - c_i, L)$$

is a non-trivial factor of L. Therefore,

$$\partial^2 = \partial \cdot \partial = \left(\partial + \frac{1}{x}\right) \cdot \left(\partial - \frac{1}{x}\right).$$

4.2 Factorization via Newton Polygons

The Newton polygon is a tool for understanding the behavior of polynomials over local fields. In the original case, as we have seen in the introduction of this chapter, the local field of $\mathbb{C}[[x]]$ is its field of fractions $\mathbb{C}((x))$. The Newton polygon is an effective device for understanding the leading terms of

$$y(x) = \sum_{j \geq d} a_j x^{j/n}$$

of the Puiseux series expansion solutions to equations

$$f(x, y(x)) = 0,$$

where $f \in \mathbb{C}[[x, y]]$; that is, implicitly defined algebraic functions. The exponents j/n depending on the chosen branch, and the solutions themselves are power series in $\mathbb{C}[[x^{1/n}]]$ for a denominator n corresponding to the branch. The Newton polygon gives an effective, algorithmic approach to calculating n and hence the y-roots of f.

In this section we present the use of the Newton polygons for factoring linear differential operators, following the exposition of van der Put and Singer [2003], which is due to the works of Malgrange [1979] and Ramis [1978].

Let k be a field of characteristic 0. A typical non-zero element $a(x) \in k((x))$ can be written as

$$a(x) = x^m \sum_{j \geq 0} a_j x^j = \sum_{j \geq m} a_{j-m} x^j \text{ where } a_0 \neq 0 \text{ and } m \in \mathbb{Z}.$$

The order of a, denoted by $\operatorname{ord}(a)$, is the exponent m of the first non-vanishing term of a. By definition the $\operatorname{ord}(0) = \infty$.

Once again we consider $k((x))$ as a differential field equipped with derivation $' = d/dx$. The ring $k((x))[\partial]$ is governed by the rule

$$\partial x = x\partial + 1.$$

Consider a new indeterminate $\delta := x\partial$, actually $\delta \in k((x))[\partial]$, and denote $k((x))[\delta]$ as the skew ring of linear differential operators in the indeterminate δ. Then, we have

$$\delta x = x\delta + x \text{ in } k((x))[\delta].$$

By the isomorphism $k((x))[\delta] \cong k((x))[\partial]$ which sends δ to $x\partial$ and any other arbitrary operator

$$\sum_i a_i \delta^i \mapsto \sum_i a_i (x\partial)^i,$$

we can represent differential operators in the form

$$L = a_n\delta^n + \cdots + a_0\delta^0 \text{ with } a_n \neq 0. \tag{4.2.1}$$

This form has several advantages, in particular if

$$a_i = \sum_{j>>-\infty} a_{ji}x^j \text{ for all } i,$$

then by the isomorphism $k((x))[\delta] \cong k[\delta]((x))$

$$L = \sum_{i=0}^{n} \left(\sum_{j>>-\infty} a_{ji}x^j \right) \delta^i = \sum_{0\leq i\leq n, j>>-\infty} a_{ji}x^j\delta^i = \sum_{j>>-\infty} x^j L(j)(\delta).$$

where

$$L(j)(\delta) = \sum_{i=0}^{n} a_{ji}\delta^i \in k[\delta] \text{ for } j >> -\infty$$

are polynomials in δ of degree bounded by the ord(L). With this we can obtain later the multiplication formula

$$\left(\sum_i x^i L_1(i)(\delta) \right) \left(\sum_j x^j L_2(j)(\delta) \right) = \sum_m x^m \sum_{i+j=m} L_1(i)(\delta + j)L_2(j)(\delta).$$

Roughly speaking, we can construct the Newton polygon $N(L)$ of f in the following way:

The **Newton polygon** $N(L)$ of an operator L is the convex hull of the union of all rectangles with vertices $(0, j), (i, j), (i, \infty)$ and $(0, \infty)$, for all points (i, j) for which $x^j\delta^i$ has a non-zero coefficient in L (i.e., the minimal convex set containing all these rectangles)".

Now, we are going to formalize this construction, in order to do this we need some basic definitions like polyhedral sets and Minkowski sum, and certain ordering on the points of the plane $\mathbb{R}^2$.

A **polyhedral set** is a subset of $\mathbb{R}^2$ that is the intersection of a finite number of closed half-planes. For practical reasons, we will consider connected polyhedral sets with non-empty interior.

The boundary of such a set is the union of a finite number of closed line segments called edges. The endpoints of the edges are called vertices or extremal points.

The vertices and the edges of a polyhedral set are collectively referred to as the faces of the set.

Given two subsets M_1 and M_2 in $\mathbb{R}^2$ the **Minkowski sum** of these two sets is the result of adding every element of M_1 to every element of M_2, i.e., the set

$$M_1 + M_2 = \{ m_1 + m_2 \mid m_1 \in M_1, m_2 \in M_2 \}.$$

In particular, any face of the sum $M_1 + M_2$ is the sum of the faces of M_1 and M_2. The same is true for vertices.

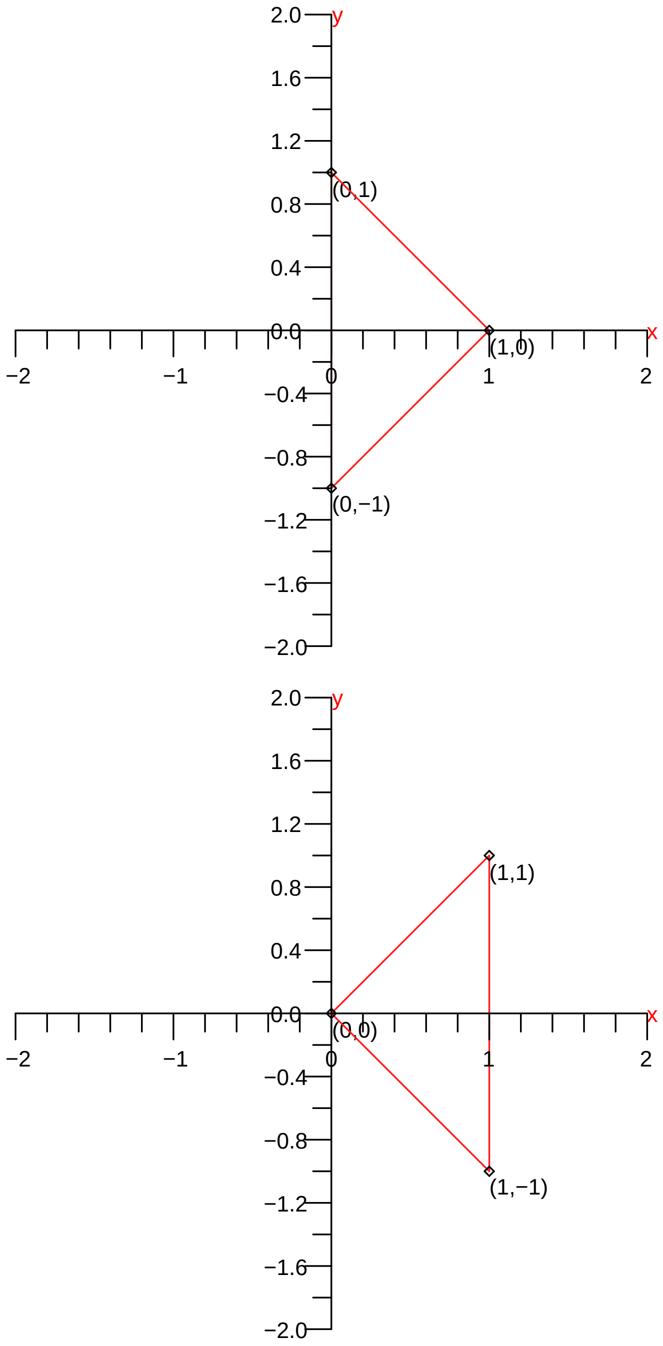

Fig. 4.1: Sets M_1 and M_2, resp.

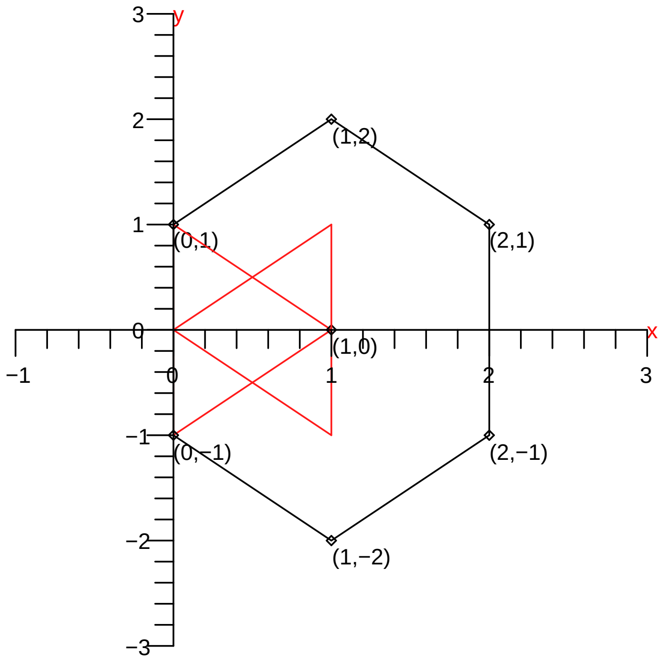

Fig. 4.2: $M_1 + M_2$

For example, if we have two 2-simplices (triangles) in $\mathbb{R}^2$, with points represented by

$$M_1 = \{ (1,0), (0,1), (0,-1) \} \text{ and } M_2 = \{ (0,0), (1,1), (1,-1) \},$$

as we have in Figure (4.2), then their Minkowski sum is

$$M_1 + M_2 = \{ (1,0), (2,1), (2,-1), (0,1), (1,2), (1,0), (0,-1), (1,0), (1,-2) \}.$$

as we can see in Figure (4.2).

Define the following partial order in $\mathbb{R}^2$

$$(x_1, y_1) \geq (x_2, y_2) \iff y_1 \geq y_2 \text{ and } x_1 \leq x_2.$$

The **monomials** in $\mathcal{D} = k((x))[\delta]$ are the elements of the form $x^m \delta^n$. The **Newton polygon** $N(L)$ of $L \neq 0$ is the convex hull of the set

$$\{ (x, y) \in \mathbb{R}^2 \mid \text{there is } x^m \delta^n \text{ in } L \text{ with } (x, y) \geq (n, m) \}.$$

$N(L)$ has finitely many extremal points

$$\{ (n_1, m_1), \ldots, (n_{r+1}, m_{r+1}) \}$$

with

$$0 \leq n_1 \leq n_2 \leq \cdots \leq n_{r+1} = n.$$

The positive **slopes** of L are

$$\mu_1 < \cdots < \mu_r \text{ with } \mu_i = \frac{m_{i+1} - m_i}{n_{i+1} - n_i} \text{ and } \mu_{r+1} = \infty.$$

If $n_1 > 0$ then one adds a slope μ_0 and in this case we put $n_0 = 0$. This definition has the property that all the slopes are ≥ 0.

If L has only one slope $\mu = 0$ then L is called **regular singular**.

Let $f : [0, n] \longrightarrow \mathbb{R}$ be given by

1. $f(n_0) = f(n_1) = m_1$.
2. $f(n_i) = m_i$ for all i.
3. f is (affine) linear on each segment $[n_i, n_{i+1}]$.

The slopes are the slopes of the graph. The length of the slope k_i is $n_{i+1} - n_i$. We reserve the term special polygon for a convex set that is the Newton polygon of some differential operator.

Let $b(L)$ denote the graph of f. The boundary part $B(L)$ of L is

$$B(L) = \sum_{(n,m)\in b(L)} a_{n,m} z^m \delta^n.$$

Write

$$L = B(L) + R(L).$$

We say that $L_1 > L_2$ if the points of $b(L_1)$ either lie in the interior of $N(L_2)$ or on the vertical ray

$$\{ (n_{r+1}, y) \mid y > m_{r+1} \}.$$

Clearly $R(L) > B(L)$ and $R(L) > L$. We note that the product of two monomials

$$M_1 := x^{m_1}\delta^{n_1} \text{ and } M_2 := x^{m_2}\delta^{n_2}$$

with $m_1, m_2 \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{N}$ is not a monomial. In fact, the product is

$$x^{m_1+m_2}(\delta + m_2)^{n_1}\delta^{n_2}. \tag{4.2.2}$$

However,

$$B(M_1M_2) = x^{m_1+m_2}\delta^{n_1+n_2}.$$

As consequence of Formula (4.2.2) we have the formula

$$x^{-i}L(j)(\delta)x^i = L(j)(\delta + i) \text{ for all } i \geq 0 \text{ and } j >> -\infty. \tag{4.2.3}$$

Form (4.2.1) has several advantages, because if $L_1, L_2 \in \mathcal{D}$ where

$$L_1 = \sum_{i>>-\infty} x^i L_1(i)(\delta) \text{ and } L_2 = \sum_{j>>-\infty} x^j L_2(j)(\delta),$$

by Formula (4.2.3) we get the multiplication formula

$$\left(\sum_{i>>-\infty} x^i L_1(i)(\delta)\right)\left(\sum_{j>>-\infty} x^j L_2(j)(\delta)\right) = \sum_{m>>-\infty} x^m \sum_{i+j=m} L_1(i)(\delta + j)L_2(j)(\delta). \tag{4.2.4}$$

The main properties of the Newton polygons are

- $N(L_1L_2) = N(L_1) + N(L_2)$,
- the set of slopes of L_1L_2 is the union of the sets of slopes of L_1 and L_2,
- the length of the slope of L_1L_2 is the sum of length of the same slope for L_1 and L_2,

for all $L_1, L_2 \in k((x))[\delta]$.

The next theorem will provide us a way to factor linear differential operators using Newton polygons. For a proof we refer to van der Put and Singer [2003], pp. 88.

Theorem 7. *Suppose that the Newton polygon of a monic differential operator L can be written as a sum of two special polygons P_1, P_2 that have no slope in common. Then there are unique monic differential operators L_1, L_2 such that P_i is the Newton polygon of L_i and $L = L_1 L_2$. Moreover,*

$$\mathcal{D}/\mathcal{D}L \cong \mathcal{D}/\mathcal{D}L_1 \oplus \mathcal{D}/\mathcal{D}L_2.$$

We will illustrate the theorem with the next example taken from van der Put and Singer [2003].

Example 16. *Let us consider the operator $L = x\delta^2 + \delta - 1$ where $ord(L) = 2$ and $m = 0$.*

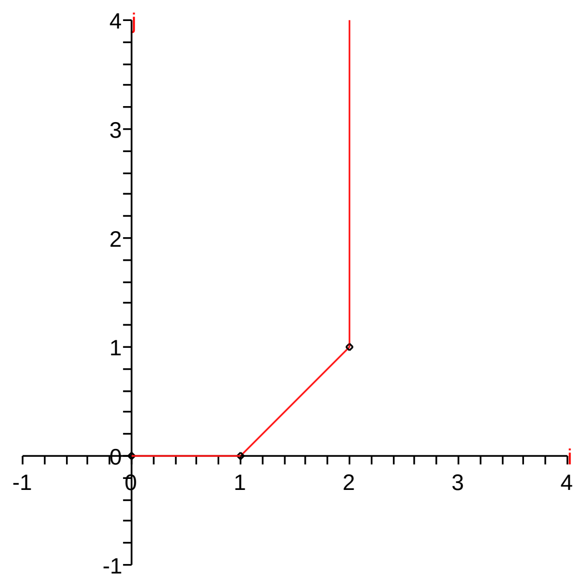

Fig. 4.3: N (L): Newton polygon of L

The Newton polygon $N(L)$ of L is represented in Figure (4.3). From this figure we can see that the Newton polygon $N(L)$ is the sum of two special polygons P_1 with unique slope 0, and P_2, with unique slope 1, namely

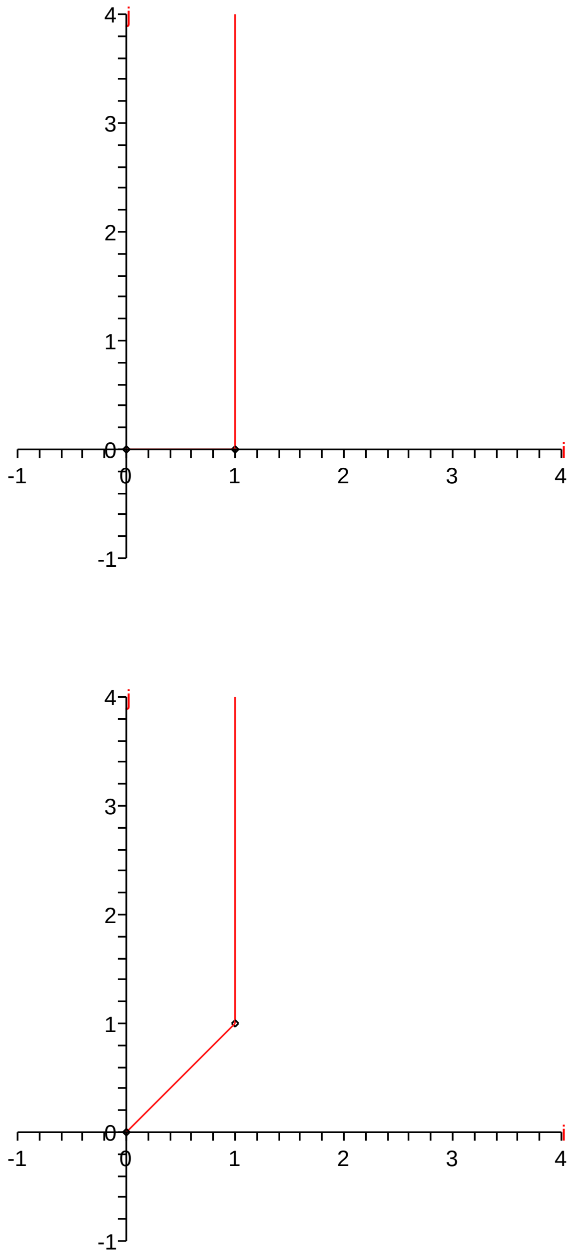

Fig. 4.4: Specials polygons P_1 and P_2, resp.

By Theorem (7) we want to find $L_1, L_2 \in k((x))[\delta]$ *such that* P_i *is the Newton polygon of* L_i *and* $L = L_1L_2$. *Suppose that*

$$L_1 = L_1(0)(\delta) + xL_1(1)(\delta) + \cdots,$$

$$L_2 = L_2(0)(\delta) + xL_2(1)(\delta) + \cdots.$$

Since $n_1 = 1$ *(i.e. the abscissa of the extremal point* $(1,0)$*) and* $ord(L)$ *we have* $L_1(0)(\delta)$ *is monic polynomial of degree* 1 *and the* $L_1(i)(\delta)$ *have degree* 0 *for* $i > 0$.

On the other hand, P_2 *has no slope equal to* 0*, this means that* $L_2(0)(\delta)$ *is a constant. In fact* $L_2(0)(\delta) = 1$, *because* $(0,0)$ *is an extremal point of* P_2. *Comparing the coefficients of* x^0 *in* $L = L_1L_2$ *we get*

$$L_1(0)(\delta)L_2(0)(\delta) = L(0)(\delta) \implies L_1(0)(\delta) = \delta - 1.$$

Comparing the coefficients of x^1 *we have*

$$L_1(0)(\delta + 1)L_2(1)(\delta) + L_1(0)(\delta)L_2(0)(\delta) = L(1)(\delta) \Rightarrow$$

$$\delta L_2(1)(\delta) + L_1(1)(\delta) = \delta^2.$$

This implies that $L_2(1)(\delta) = \delta$ *and* $L_1(1)(\delta) = 0$. *It can be shown by induction that*

$$L_1(i)(\delta) = L_2(i)(\delta) = 0 \textit{ for } i \geq 2.$$

Therefore, the operator L *factors as* $L = L_1L_2$ *where*

$$L_1 = \delta - 1 \textit{ and } L_2 = x\delta + 1.$$

4.3 Factorization via Newton Polynomials and Coprime Index 1

From Malgrange [1979] we know that an element of $k((x))[\partial]$ which has only one slope in the Newton polygon and which has an irreducible Newton polynomial is irreducible in $k((x))[\partial]$. He shows that in the following two cases a differential operator $L \in k((x))[\partial]$ is reducible in this ring and how a factorization can be computed:

1. An operator with broken Newton polygon (i.e. more than one slope).
2. An operator with one slope > 0 where the Newton polynomial is reducible and not a power of an irreducible polynomial.

In van Hoeij [1997a] these two cases of factorization and the factorization of regular singular operators are unified in the so called coprime index 1 factorization.

Since the elements of $k((x))$ consist of infinitely many terms, only a finite number of them can be computed. This means that a factorization can only be determinated up to some finite accuracy.

We start this section with some basic definitions like filtered ring, filtration and associated graded ring. These important concepts will become very useful, since often information can be obtained by passing from a ring with a "natural" filtration to the associated graded ring, and then translating the result back to the original ring. In particular, this is useful if the ring is filtered by (additive) subgroups, such that the associated graded ring is commutative.

A **filtration** of a ring D is a chain of additive subgroups D_i of D, such that

$$1 \in D_0, \ \cdots \supset D_{i-1} \supset D_i \supset D_{i+1} \supset \cdots, \ i \in \mathbb{Z}, \text{with}$$

$$D_iD_j \subset D_{i+j} \text{ for all } i, j \in \mathbb{Z}, \text{ and } D = \bigcup_{i \in \mathbb{Z}} D_i.$$

A ring equipped with a filtration is called a **filtered ring**.

A **grading** $\{D_i\}_{i\in\mathbb{Z}}$ of the ring D is a sequence of additive subgroups D_i of D such that

$$D = \bigoplus_{i\in\mathbb{Z}} D_i \text{ and } D_i D_j \subseteq D_{i+j} \text{ for all } i, j \in \mathbb{Z}.$$

A ring with a grading $\{D_i\}$ is called **graded**. The elements of D_i are called **homogeneous of degree** i. A homogeneous element of R is simply an element of one of the groups D_i. If $f \in D$, there is a unique expression for f of the form

$$f = f_0 + f_1 + \cdots \text{ with } f_i \in D_i \text{ and } f_j = 0 \text{ for } j >> 0,$$

the f_i are called the **homogeneous components** of f. In particular, the component D_0 is a subring of D containing 1_D.

Let D be a filtered ring with a filtration $\{D_i\}_{i\in\mathbb{Z}}$. The **associated graded ring**, denoted by $gr(D)$ is defined as

$$gr(D) = \bigoplus_{i\in\mathbb{Z}} D_i/D_{i+1} \text{ (as additive groups)},$$

equipped with the obvious addition and multiplication given by

$$(r + D_{i+1})(s + D_{j+1}) = (rs + D_{i+j+1}), \text{ for } r \in D_i, s \in D_j.$$

The symbol map $\sigma : D \to gr(D)$ is defined as:

$$\sigma(0) = 0, \text{ and } \sigma(f) = f + D_{i+1} \text{ if } f \in D_i \backslash D_{i+1}.$$

Let D be a ring, a **discrete valuation** on the ring D is a map $v : D \to \mathbb{Z} \cup \{\infty\}$ such that

- $v(fg) = v(f) + v(g)$,
- $v(f + g) \geq \min(v(f), v(g))$, and
- $v(0) = \infty$,

for all $f, g \in D$. As a consequence of the first two properties we have

$$v(f + g) = \min(v(f), v(g)) \text{ if } v(f) \neq v(g).$$

A valuation v defines a filtration on a ring D as follows

$$D_i = \{ f \in D \,|\, v(f) \geq i \}.$$

For a positive integer a the set D_0/D_a has the structure of a ring.

For a ring D with a valuation v we define a **truncation** σ_a with **accuracy** a for non-zero elements $f \in D$ and a positive integer a as follows

$$\sigma_a(f) = f + D_{v(f)+a} \in D_{v(f)}/D_{v(f)+a}.$$

Let k be a field of characteristic 0 and consider the ring $\mathcal{D} = k((x))[\delta]$, where $\delta = x\partial \in k((x))[\partial]$. Let $L \in \mathcal{D}$ be a differential operator given by

$$L = \sum_{j,i} a_{ji} x^j \delta^i.$$

Let $s \in \mathbb{Q}$ be a rational number with $s \geq 0$, write $s = u/d$ with $u, d \in \mathbb{Z}$, $\gcd(u, d) = 1$ and $d > 0$. Then the function $v_s : \mathcal{D} \to \mathbb{Z} \cup \{\infty\}$ given by

$$v_s\left(\sum_{j,i} a_{ji} x^j \delta^i\right) = \inf\{jd - iu : a_{ji} \neq 0\},$$

defines a discrete valuation on $\mathcal{D}$. We take infimum instead of minimum for formal reasons with respect to the zero operator. As mentioned above, the valuation v_s gives a filtration $(D_i)_{i\in\mathbb{Z}}$

$$D_i = \{L \in \mathcal{D} | v_s(L) \geq i\}$$

and a truncation $\sigma_{a,s}$ with accuracy a

$$\sigma_{a,s}(L) = L + D_{v_s(L)+a} \in D_{v_s(L)}/D_{v_s(L)+a}.$$

To $\sigma_{1,s}$ for $L \in \mathcal{D}$ corresponds the so-called **Newton polynomial** $N_s(L)$ of L for slope s (which does not correspond to the usual definition of the Newton polynomial), and roughly speaking we can compute it in the following way:

"The length $l(s)$ of a slope s in the Newton polygon $N(L)$ is defined as the length of the projection of this slope onto the x-axis. The Newton polynomial $N_s(L)$ is a polynomial in a new indeterminate T of degree

$$\frac{l(s)}{d},$$

and its monomials can be computed from the points which lie exactly on the slope s and the leading coefficients of the corresponding a_{ji}".

In particular, $\sigma_{1,s}(L)$ is an element of

$$\overline{D} = \bigcup_{i\in\mathbb{Z}} D_i/D_{i+1}.$$

A multiplication is defined for elements of $\overline{D}$, and an addition is only defined for $f, g \in \overline{D}$ which are elements of the same D_i/D_{i+1}. There is a k-linear bijection $N'_s : D_0/D_1 \to k[T]$ which is also a ring isomorphism if $i = 0$. If $i = 0$ then N'_0 is defined by

$$x^u\delta^d \longmapsto T.$$

For every $i \in \mathbb{Z}$ there is a unique pair of integers u_i, d_i with

$$0 \leq d_i < d \text{ and } v_s(x^{u_i}\delta^{d_i}) = i$$

such that the map $\phi_i : D_0/D_1 \to D_i/D_{i+1}$ defined by

$$\phi_i(a) = x^{u_i}\delta^{d_i}a$$

is a bijection. Now, for $s > 0$ let $N'_s : D_i/D_{i+1} \to k[T]$ be defined by

$$N'_s(a) = N'_s\left(\phi_i^{-1}(a)\right)$$

as we see in the next diagram

$$\begin{array}{ccc} D_0/D_1 & \xrightarrow{N'_s} & k[T] \\ \Big\downarrow \phi_i & \nearrow_{N'_s} & \\ D_i/D_{i+1} & & \end{array}.$$

N'_s is also defined for non-zero elements of $L \in \mathcal{D}$ as $N'_s(\sigma_{1,s}(L))$. For slope $s = 0$ define the **Newton polynomial** $N_0(L)$ as $N'_0(L)$. By Formula (4.2.4) the following property follows for $Q, R \in \mathcal{D}$

$$N_0(QR) = S_{T=T+v_0(R)}(N_0(Q))N_0(R).$$

Here $S_{T=T+v_0(R)}(N_0(Q))$ means $N_0(Q)$ with T replaced by $T + v_0(R)$. For slope $s > 0$ we have the following property for $Q, R \in \mathcal{D}$

$$N'_s(QR) = T^pN'_s(Q)N'_s(R).$$

Here p is either 0 or 1, depending on the slope s and the valuations $v_s(Q)$ and $v_s(R)$. Let $i = v_s(Q)$ and $j = v_s(R)$. Then

$$\phi_i(1) \cdot \phi_j(1) = x^{u_i+u_j}\delta^{d_i+d_j} \mod D_{i+j+1}.$$

This is either equal to $\phi_{i+j}(1)$ or $x^u\delta^d\phi_{i+j}(1) \mod D_{i+j+1}$, depending on whether d_i+d_j is smaller than d or not. In the first case $p = 0$, in the second case $p = 1$.

Now, for $s > 0$ define the **Newton polynomial** $N_s(L)$ of L for slope $s > 0$ as $N'_s(L)$ divided by T to the power the multiplicity of the factor T in $N'_s(L)$. Then

$$N_s(QR) = N_s(Q)N_s(R)$$

for $s > 0$ and for all $Q, R \in \mathcal{D}$. If $s > 0$ then

$$\sigma_{1,s}(Q)\sigma_{1,s}(R) = \sigma_{1,s}(QR) = \sigma_{1,s}(R)\sigma_{1,s}(Q)$$

for all Q and R in $\mathcal{D}$. If $s = 0$ then

$$\sigma_{1,s}(Q)\sigma_{1,s}(R) = \sigma_{1,s}(QR) = S_{-v_s(Q)}\left(\sigma_{1,s}(R)\right) \cdot S_{v_s(R)}\left(\sigma_{1,s}(Q)\right).$$

So $\sigma_{1,s}$ is commutative (i.e. is the same for QR and RQ) if $s > 0$. If $s = 0$ then $\sigma_{1,s}$ is commutative up to substitutions $S_{-v_s(Q)}$ and $S_{v_s(R)}$ which map δ to δ plus some integer.

The Newton polynomial is useful for factorization in $\mathcal{D}$ because if $L = QR$ then $\sigma_{1,s}(Q)\sigma_{1,s}(R) = \sigma_{1,s}(L)$. So a factorization of L induces a factorization of the Newton polynomial.

The roots of $N_0(L)$ in $\overline{k}$ are called the **exponents** of L. If $L \in \mathcal{D}$ is regular singular (i.e. L has only one slope $s = 0$, or equivalently $\deg(N_0(L)) = \mathrm{ord}(L)$) and all exponents of L are integers, then L is called **semi-regular**.

Property: If $L = QR$ then the Newton polynomial of the right-hand factor $N_s(R)$ divides $N_s(L)$. However, for a left-hand factor this need not hold. But if $s > 0$ or if $v_0(R) = 0$ (for example if R is regular singular and monic) then

$$N_s(L) = N_s(Q)N_s(R)$$

so in such cases $N_s(Q)$ divides $N_s(L)$.

Example 17. *Consider the operator*

$$L = \delta^3 + \left(\frac{2x+1}{x^2}\right)\delta^2 - \left(\frac{2x^2-x-1}{x^3}\right)\delta + \left(\frac{x^2-x-1}{x^3}\right) =$$

$$-x^{-3} - x^{-2} + x^{-1} + x^{-3}\delta + x^{-2}\delta - 2x^{-1}\delta + x^{-2}\delta^2 + 2x^{-1}\delta^2 + \delta^3 =$$

$$x^{-3}(-1+\delta) + x^{-2}(-1+\delta+\delta^2) + x^{-1}(1-2\delta+\delta^2) + \delta^3.$$

The Newton polygon $N(L)$ of L is given in Figure (4.5), where the slopes are $s = 0, 1,$ and 2. We compute the Newton polynomial $N_s(L)$ of L for the slope s:

1. *Compute the $\deg(N_s(L))$ using $l(s)$ the length of the slope s.*
2. *Compute the monomials using the leading coefficient of the points which lie exactly on the slope s.*

- *For $s = 0$ we get*

$$\deg(N_0(L)) = \frac{l(s)}{1} = \frac{l(0)}{1} = \frac{1}{1} = 1$$

and for the points $(0, -3)$ and $(1, -3)$ we have $a_{0,-3} = -1$, $a_{1,-5} = 1$. Then,

$$N_0(L) = a_{0,-3} + a_{1,-3}\,T = -1 + T.$$

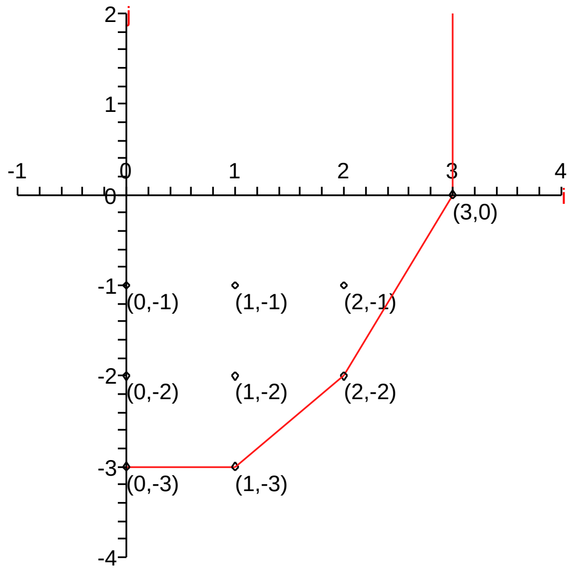

Fig. 4.5: Newton polygon of L

- *For* $s = 1$ *we have*
$$\deg(N_1(L)) = \frac{l(1)}{1} = \frac{1}{1} = 1$$
and for the points $(1, -3)$ *and* $(2, -2)$ *we have* $a_{1,-3} = 1$*, and* $a_{2,-2} = 1$*. Then,*
$$N_1(L) = a_{1,-3} + a_{2,-2}T = 1 + T.$$

- *For* $s = 2$ *we get*
$$\deg(N_2(L)) = \frac{l(2)}{1} = \frac{1}{1} = 1$$
and for the points $(2, -2)$ *and* $(3, 0)$ *we have* $a_{2,-2} = 1$ *and* $a_{3,0} = 1$*. Then,*
$$N_2(L) = a_{2,-2} + a_{3,0}T = 1 + T.$$

Now let us compute the Newton polynomial in the formal way using the valuation v_s *for each of the slopes. For* $s = 0$ *we have* $u = 0$*,* $d = 1$*, whence*

$$v_0(L) = \inf(-3.1 - 0.0, -2.1 - 0.0, -1.1 - 0.0, -3.1 - 1.0,$$
$$-2.1 - 1.0, -2.1 - 1.0, -2.1 - 2.0, -1.1 - 2.0, 0.1 - 3.0) =$$
$$\inf(-3, -2, -1, -3, -2, -1, -2, -1, 0) = -3.$$
$$v_0(L) = -3.$$

For $s = \frac{1}{2}$ *we have* $u = 1$ *and* $d = 2$*, whence*
$$v_1(L) = \inf(-3, -2, -1, -4, -3, -2, -4, -3, -3) = -4.$$
$$v_1(L) = -4.$$

For $s = 2$ *we have* $u = 2$ *and* $d = 1$*, whence*
$$v_2(L) = \inf(-3, -2, -1, -5, -4, -3, -6, -5, -6) = -6$$

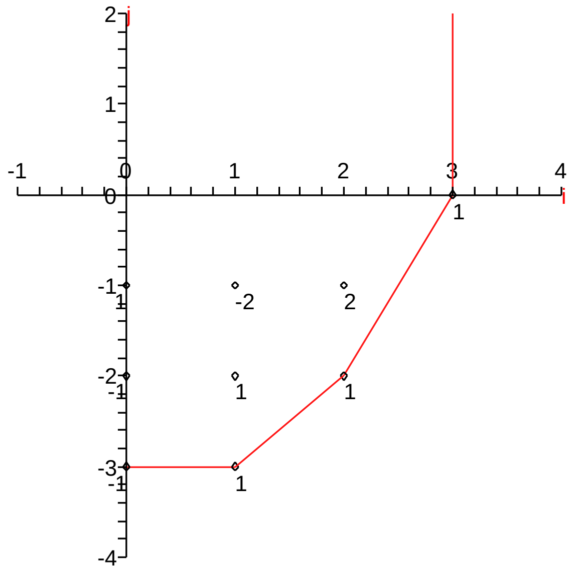

Fig. 4.6: Newton polygon of L

$$v_2(L) = -6.$$

On the other hand, the truncation for each of the slopes are

$$\sigma_{1,0}(L) = L + D_{v_0(L)+1} \in D_{v_0(L)}/D_{v_0(L)+1} \;\Rightarrow\; \sigma_{1,0}(L) = L + D_{-3} \in D_{-3}/D_{-2}.$$

Similarly,

$$\sigma_{1,1}(L) = L + D_{-4} \in D_{-4}/D_{-3}, \text{ and } \sigma_{1,2}(L) = L + D_{-6} \in D_{-6}/D_{-5}.$$

Now, we need to compute the polynomials $N'_s(L)$ for all the slopes,

- *For $s = 0$ the bijection $N'_0 : D_0/D_1 \to k[T]$ is defined by*

$$N'_0(x^0\delta^1) = N'_0(\delta) = T.$$

Now, for $i = v_0(L) = -3$ and $d = 1$ we have $0 \le d_i < 1$ and $v_0(x^{u_i}\delta^{d_i}) = -3$ that means

$$d_i = 0 \;\text{ and }\; u_1.1 - d_i.0 = -3 \;\Rightarrow\; u_i = -3.$$

Let $\phi_{-3} : D_0/D_1 \to D_{-3}/D_{-2}$ be defined as

$$\phi_{-3}(a) = x^{-3}\delta^0 a = x^{-3}a$$

whence

$$N'_0(L) = N'_0\left(\sigma_{1,0}(L)\right) = N'_0(L + D_{-3}) = N'_0(-x^{-3} + x^{-3}\delta) =$$
$$-N'_0(x^{-3}) + N'_0(x^{-3}\delta) = -N'_0\left(\phi_{-3}^{-1}(x)\right) + N'_0\left(\phi_{-3}^{-1}(x\delta)\right) =$$
$$-N'_0(1) + N'_0(\delta) = -1 + T.$$

Therefore,

$$N'_0(L) = -1 + T \;\Rightarrow\; N_0(L) = 1 + T.$$

- *For $s=1$ the bijection $N_1' : D_0/D_1 \to k[T]$ is defined by*

$$N_1'(x^1\delta^1) = T.$$

For $i = v_1(L) = -4$ and $d = 1$ we have

$$0 \le d_{-4} < 1 \quad \text{and} \quad v_1(x^{u_{-4}}\delta^{d_{-4}}) = -4$$

that means

$$d_{-4} = 0 \text{ and } u_{-4}.1 - d_{-4}.1 = -4 \;\Rightarrow\; u_{-13} = -4.$$

Let $\phi_{-4} : D_0/D_1 \to D_{-4}/D_{-3}$ be defined by

$$\phi_{-4}(a) = x^{-4}\delta^0 a = x^{-4}a$$

whence

$$N_1'(L) = N_1'(\sigma_{1,1}(L)) = N_1'(L + D_{-4}) =$$

$$N_1'(x^{-3}\delta + x^{-2}\delta^2) = N_1'\left(\phi_{-4}^{-1}(x\delta)\right) + N_1'\left(\phi_{-4}^{-1}(x^2\delta^2)\right) =$$

$$N_1'(x\delta) + N_1'(x^2\delta^2) = T + T^2 = T(1+T)$$

Therefore,

$$N_1(L) = N_1'(L)/T = 1 + T.$$

- *For $s=2$ the bijection $N_2' : D_0/D_1 \to k[T]$ is defined by*

$$N_2'(x^2\delta^1) = T$$

For $i = v_2(L) = -6$ and $d = 1$ we have

$$0 \le d_{-6} < 1 \quad \text{and} \quad v_1(x^{u_{-6}}\delta^{d_{-6}}) = -6$$

that means

$$d_{-6} = 0 \quad \text{and} \quad u_{-6}1 - d_{-6}.1 = -6 \;\Rightarrow\; u_{-6} = -6$$

Let $\phi_{-6} : D_0/D_1 \to D_{-6}/D_{-5}$ be defined as

$$\phi_{-6}(a) = x^{-6}\delta^0 a = x^{-6}a$$

whence

$$N_2'(L) = N_2'(\sigma_{1,1}(L)) = N_2'(L + D_{-6}) = N_2'(x^{-2}\delta^2 + \delta^3) =$$

$$N_2'(x^{-2}\delta^2) + N_2'(\delta^3) = N_2'\left(\phi_{-6}^{-1}(x^4\delta^2)\right) + N_2'\left(\phi_{-6}^{-1}(x^6\delta^3)\right) =$$

$$N_2'(x^4\delta^2) + N_2'(x^6\delta^3) = T^2 + T^3 = T^2(1+T).$$

Therefore,

$$N_2(L) = N_2'(L)/T^2 = 1 + T.$$

Suppose that $L \in k((x))[\delta]$ is monic and $L = QR$ is a non-trivial factorization, where Q and R are monic elements of $k((x))[\delta]$. The **coprime index** of a factorization $L = QR$ is the smallest positive integer t for which the following holds:

For all integers $a \geq t$ and monic elements $\widetilde{Q}$ and $\widetilde{R}$ of $\mathcal{D}$, if

$$\sigma_{a,s}(\widetilde{Q}) = \sigma_{a,s}(Q) \text{ and } \sigma_{a,s}(\widetilde{R}) = \sigma_{a,s}(R) \text{ and } \sigma_{a+t,s}(\widetilde{Q}\widetilde{R}) = \sigma_{a+t,s}(L)$$

then

$$\sigma_{a+1,s}(\widetilde{Q}) = \sigma_{a+1,s}(Q) \text{ and } \sigma_{a+1,s}(\widetilde{R}) = \sigma_{a+1,s}(R).$$

Coprime index 1 means that $\gcd(Q, R) = 1$ and then the factorization can be lifted by the usual Hensel lifting algorithm. In this case we must solve a system of the form

$$l\sigma_{1,s}(R) + \sigma_{1,s}(Q)r = g$$

where g is computed by multiplying the so far computed truncations (called $\widetilde{Q}$ and $\widetilde{R}$) of Q and R and subtracting this product from L. This equation can be converted to an equation

$$qR_0 + rQ_0 = g$$

for certain $q, r, Q_0, R_0, g \in k[T]$ and q, r unknown. Such an equation can be solved by the Euclidean algorithm.

Now the question is:

How to compute $\sigma_{a+1,s}(Q)$ and $\sigma_{a+1,s}(R)$ from $\sigma_{a,s}(Q), \sigma_{a,s}(R)$ and L?

Suppose that $t \leq a$, we will use indeterminates for those coefficients of $\sigma_{a+t,s}(Q)$ and $\sigma_{a+t,s}(R)$ which are not yet known. Then the equation

$$\sigma_{a+t,s}(QR) = \sigma_{a+t,s}(L)$$

gives a set of equations in these unknowns. $t \leq a$ is needed to ensure that all the equations are linear.

Coprime index t means that $\sigma_{a+t,s}(Q)$ and $\sigma_{a+t,s}(R)$ can be uniquely determined from these linear equations. A truncation

$$\sigma_{a,s}(R) = R + D_{v_s(R)+a}$$

is store as an element $\widetilde{R} \in k[x, 1/x, \delta]$ with no terms in $D_{v_s(R)+a}$. Now write

$$r = \sum_{j,i} r_{ji} x^j \delta^i$$

where the sum is taken over all j, i such that

$$v_s(R) + a \leq v_s(x^j\delta^i) < v_s(R) + a + t \quad \text{and} \quad i \leq \operatorname{ord}(R).$$

Here $r_{j,i}$ are indeterminates. We set

$$r_{ji} = 0 \text{ for } i = \operatorname{ord}(R), \text{ and } j \neq 0$$

and set

$$r_{ji} = 1 \text{ for } i = \operatorname{ord}(R), \text{ and } j = 0.$$

Similarly write $\widetilde{Q}$ and q. Now look for values for the l_{ji} and r_{ji} such that

$$\widetilde{R} + r \text{ and } \widetilde{Q} + q$$

approximate R and Q up to accuracy $a + 1$. If the coprime index is t, the accuracy is at least $a + 1$ if the following holds:

$$\sigma_{a+t,s}\Big((\widetilde{Q} + q)(\widetilde{R} + r)\Big) = \sigma_{a+t,s}(L),$$

or equivalently

$$(\widetilde{Q}+q)(\widetilde{R}+r) \equiv L \mod D_{v_s(L)+a+t}$$

$$(\widetilde{Q}+q)(\widetilde{R}+r) = \widetilde{Q}\widetilde{R} + qR' + \widetilde{Q}r + qr.$$

To determine

$$q\widetilde{R} \mod D_{v_s(L)+a+t}$$

it suffices to have $\widetilde{R}$ up to accuracy t because

$$v_s(q) + v_s(\widetilde{R}) \geq v_s(L) + a.$$

Similarly $v_{t,s}(\widetilde{Q})$ suffices to compute

$$\widetilde{Q}r \mod D_{v_s(L)+a+t}.$$

$$v_s(qr) \geq v_s(L) + a + a \geq v_s(L) + a + t$$

so qr vanishes modulo $D_{v_s(L)+a+t}$. Hence

$$L \equiv \widetilde{Q}\widetilde{R} + q\sigma_{t,s}(\widetilde{R}) + \sigma_{t,s}(\widetilde{Q})r \mod D_{v_s(L)+a+t}.$$

By equating the coefficients of the left-hand side to the coefficients of the right-hand side (the coefficients of all monomials of valuation $< v_s(L) + a + t$) we find the linear equation in q_{ji} and r_{ji}. To determine these equations we must multiply q by $\sigma(\widetilde{R})$, ($= \sigma_{t,s}(R)$ because $\widetilde{R}$ equals R up to the accuracy a and $t \leq a$) which is the lowest block of R with slope s and with t in the Newton polygon of R. Similarly we must compute $\sigma_{t,s}(\widetilde{Q})r$.

Algorithm Coprime Index 1 Factorizations

Input: $L \in k((x))[\delta]$, L monic

Output: All monic coprime index 1 factorizations $L = QR$ in $k((x))[\delta]$ such that does not have a non-trivial coprime index 1 factorization.

Note: The definition of coprime index 1 depends on the valuation that is chosen on $k((x))[\delta]$.

for all slopes of L do

$g := N_s(L)$
Compute a prime factorization of g in $k[T]$, $g = cg_1^{e_1} \dots g_r^{e_r}$,
where the g_i are the different monic irreducible factors and $c \in k$.
if $s = 0$ **then**
$M := \{g_1, \dots, g_r\}$
$N := M \setminus \{ g \in M \mid \exists h \in M \wedge i \in \mathbb{N}_+ \text{ s. t. } g(T) = h(T+i) \}$
else
$N := \{g_1^{e_1}, \dots, g_r^{e_r}\}$
end if
for h in N **do**
Write $h = T^p + h_{p-1}T^{p-1} + \dots + h_0T^0$.
Write $s = u/d$ with $d > 0$ and $\gcd(u, d) = 1$
(if $s = 0$ then $u = 0$, $d = 1$)

$$\widetilde{R} := \delta^{pd} + h_{p-1}x^{-n}\delta^{(p-1)d} + h_{p-2}x^{-2n}\delta^{(p-2)d} + \dots + h_0x^{-pn}\delta^0. \quad (4.3.5)$$

Now $\widetilde{R}$ has Newton polynomial h. We want to lift $\widetilde{R}$ to a right-hand factor R such that $\widetilde{R}$ is R modulo $D_{v_s(\widetilde{R})+1}$.

$\widetilde{Q}$:= an operator such that $\sigma_{1,s}(L) = \sigma_{1,s}(\widetilde{Q}\widetilde{R})$.

$\widetilde{Q}$ is uniquely determined by requiring that $\widetilde{Q}$ has no monomials of valuation $> v_s(\widetilde{Q})$.

$L, \widetilde{Q}, \widetilde{R}$ with the lift algorithm gives a factorization $L = QR$

end do

end do

Example 18. *Let us consider again the operator of Example 17*

$$L = -x^{-3} - x^{-2} + x^{-1} + x^{-3}\delta + x^{-2}\delta - 2x^{-1}\delta + x^{-2}\delta^2 + 2x^{-1}\delta^2 + \delta^3,$$

where the slopes of the Newton polygon are $s = 0, 1,$ and 2, and the Newton polynomials for each of the slopes are

$$N_0(L) = -1 + T,\ N_1(L) = 1 + T,\ \text{and } N_2(L) = 1 + T.$$

We want to find operators $\widetilde{R}$ which has Newton polynomial N_s and also operators $\widetilde{Q}$ such that $\sigma_{1,s}(L) = \sigma_{1,s}(\widetilde{Q}\widetilde{R})$ for each of the slopes.

- *For $s = 0$ we have $u = 0$ and $d = 1$, $N = \{T - 1\}$ with $p = 1$. By Equality 4.3.5,*

$$\widetilde{R} = \delta^1 + (-1)x^0\delta^1 = \delta - 1,$$

then

$$v_0(\widetilde{R}) = \inf(0.1 - 1.0, 0.1 - 0.0) = \inf(0, 0) = 0$$

and

$$\sigma_{1,0}(L) = \sigma_{1,0}(\widetilde{Q}\widetilde{R}) \ \Rightarrow\ L + D_{v_0(L)+1} = \widetilde{Q}\widetilde{R} + D_{v_0(\widetilde{Q}\widetilde{R})+1} \ \Rightarrow$$

$$L + D_{-3+1} = \widetilde{Q}\widetilde{R} + D_{v_0(\widetilde{Q})+v_0(\widetilde{R})+1} \ \Rightarrow\ L + D_{-2} = \widetilde{Q}\widetilde{R} + D_{v_0(\widetilde{Q})+1}.$$

In particular,

$$-2 = v_0(\widetilde{Q}) + 1 \ \Rightarrow\ v_0(\widetilde{Q}) = -3.$$

Whence,

$$L + D_{-2} = \widetilde{Q}\widetilde{R} + D_{-2} \ \Rightarrow\ -x^{-3} + x^{-3}\delta = \widetilde{Q}(\delta - 1) \ \Rightarrow$$

$$x^{-3}(-1 + \delta) = \widetilde{Q}\,(\delta - 1).$$

Thus, $\widetilde{Q} = x^{-3}$.

- *For $s = 1$ we have $u = 1$ and $d = 1$, $N = \{T + 1\}$ with $p = 1$. By Equality 4.3.5,*

$$\widetilde{R} = \delta^1 + 1x^{-1}\delta^0 = \delta + x^{-1},$$

then

$$v_1(\widetilde{R}) = \inf(-1.1 - 0.1, 0.1 - 1.1) = \inf(-1, -1) = -1$$

and

$$\sigma_{1,1}(L) = \sigma_{1,1}(\widetilde{Q}\widetilde{R}) \ \Rightarrow\ L + D_{v_1(L)+1} = \widetilde{Q}\widetilde{R} + D_{v_1(\widetilde{Q}\widetilde{R})+1} \ \Rightarrow$$

$$L + D_{-4+1} = \widetilde{Q}\widetilde{R} + D_{v_1(\widetilde{Q})+v_1(\widetilde{R})+1} \ \Rightarrow\ L + D_{-3} = \widetilde{Q}\widetilde{R} + D_{v_1(\widetilde{Q})-1+1}.$$

In particular, $v_1(\widetilde{Q}) = -3$. *Whence,*

$$L + D_{-3} = \widetilde{Q}\widetilde{R} + D_{-3} \;\Rightarrow\; x^{-3}\delta + x^{-2}\delta + x^{-2}\delta^2 = \widetilde{Q}\,(x^{-1}+\delta) \;\Rightarrow$$

$$x^{-2}\delta(x^{-1}+\delta) = \widetilde{Q}\,(x^{-1}+\delta) \;\Rightarrow\; \widetilde{Q} = x^{-2}\delta.$$

Thus, $\widetilde{Q} = x^{-2}\delta$.

- *For* $s = 2$ *we have* $u = 2$ *and* $d = 1$, $N = \{T-1\}$ *with* $p = 1$. *By Equality 4.3.5,*

$$\widetilde{R} = \delta^1 + (-1)x^{-2}\delta^0 = \delta + x^{-2},$$

then

$$v_2(\widetilde{R}) = \inf(-2.1 - 0.2, 0.1 - 1.2) = \inf(-2,-2) = -2,$$

and

$$\sigma_{1,2}(L) = \sigma_{1,2}(\widetilde{Q}\widetilde{R}) \;\Rightarrow\; L + D_{v_2(L)+1} = \widetilde{Q}\widetilde{R} + D_{v_2(\widetilde{Q}\widetilde{R})+1} \;\Rightarrow$$

$$L + D_{-6+1} = \widetilde{Q}\widetilde{R} + D_{v_2(\widetilde{Q})+v_2(\widetilde{R})+1} \;\Rightarrow\; L + D_{-5} = \widetilde{Q}\widetilde{R} + D_{v_2(\widetilde{Q})-2+1}.$$

In particular,

$$-5 = v_2(\widetilde{Q}) - 1 \;\Rightarrow\; v_2(\widetilde{Q}) = -4.$$

Whence,

$$L + D_{-5} = \widetilde{Q}\widetilde{R} + D_{-5} \;\Rightarrow\; x^{-2}\delta^2 + \delta^{-3} = \widetilde{Q}\,(x^{-2}+\delta).$$

It is easy to verify that there is no operator $\widetilde{Q}$ *with* $v_2(\widetilde{Q}) = -4$ *which can satisfy the last equality. In fact,*

$$x^{-2}\delta^2 + \delta^{-3} = (x^{-2}+\delta)\delta^2$$

but not on the left, because it is also clear by the non-commutativity of the multiplication in $\mathcal{D}$, *that*

$$(x^{-2}+\delta)\delta^2 \neq \delta^2(x^{-2}+\delta).$$

Now, we want to lift $\widetilde{Q}$ *and* $\widetilde{R}$ *to* Q *and* R. *Let* $q, r \in \mathcal{D}$ *be given by*

$$q = \sum_{j,i} q_{ji}x^j\delta^i \text{ and } r = \sum_{j,i} r_{ji}x^j\delta^i,$$

where q_{ji} *and* r_{ji} *are unknowns and the sums are taken over all* j, i *such that*

$$v_s(\widetilde{Q}) + 1 \leq v_s(x^j\delta^i) < v_s(\widetilde{Q}) + 2 \text{ and } v_s(\widetilde{R}) + 1 \leq v_s(x^j\delta^i) < v_s(\widetilde{R}) + 2 \;\Rightarrow$$

respectively. So,

$$(\widetilde{Q}+q)(\widetilde{R}+r) + D_{v_s(L)+1} = L + D_{v_s(L)+1}.$$

Then we have also

$$q_{ji} = \begin{cases} 0 & \text{for } \; i = ord(\widetilde{Q}) \text{ and } j \neq 0 \\ 1 & \text{for } \; i = ord(\widetilde{Q}) \text{ and } j = 0 \end{cases}, \quad r_{ji} = \begin{cases} 0 & \text{for } \; i = ord(\widetilde{R}) \text{ and } j \neq 0 \\ 1 & \text{for } \; i = ord(\widetilde{R}) \text{ and } j = 0 \end{cases},$$

and

$$qr \in D_{v_s(L)+1}.$$

- *For $s = 0$ we have $\widetilde{R} = \delta - 1$, $\widetilde{Q} = x^{-3}$ and for q we get*

$$v_0(\widetilde{Q}) + 1 \leq v_0(x^j\delta^i) < v_0(\widetilde{Q}) + 2 \;\Rightarrow\; -2 \leq v_0(x^j\delta^i) < -1 \;\Rightarrow$$
$$v_0(x^j\delta^i) = -2,$$

for r we get

$$v_0(\widetilde{R}) + 1 \leq v_1(x^j\delta^i) < v_1(\widetilde{R}) + 2 \;\Rightarrow\; 1 \leq v_0(x^j\delta^i) < 2 \;\Rightarrow$$
$$v_0(x^j\delta^i) = 1,$$

in turn

$$q_{02} = 1 \text{ and } q_{j2} \text{ for } j \neq 0, \text{ and } r_{01} = 1 \text{ and } r_{j1} \text{ for } j \neq 0.$$

Whence,

$$r = r_{10}x \text{ and } q = q_{-20}x^{-2} + q_{-11}x^{-1}\delta + \delta^2.$$

Substituting the product $\widetilde{Q}\widetilde{R}$ and the operator L in the equality

$$\widetilde{Q}\widetilde{R} + q\widetilde{R} + \widetilde{Q}r = L,$$

we obtain,

$$x^{-3}\delta(\delta - 1) + \widetilde{Q}r + q\widetilde{R} =$$
$$-x^{-3} - x^{-2} + x^{-1} + x^{-3}\delta + x^{-2}\delta - 2x^{-1}\delta + x^{-2}\delta^2 + 2x^{-1}\delta^2 + \delta^3.$$

After some computations and simplifications we get

$$q_{-20}x^{-2}\delta - q_{-20}x^{-2} + q_{-11}x^{-1}\delta^2 - q_{-11}x^{-1}\delta - \delta^2 + r_{10}x^{-4} =$$
$$-x^{-2} + x^{-1} + x^{-2}\delta - 2x^{-1}\delta + x^{-2}\delta^2 + 2x^{-1}\delta^2 \;\Rightarrow$$

$$\begin{cases} q_{-2,0} & = & 1 \\ q_{-1,1} & = & 2 \\ q_{1,0} & = & 0 \\ -\delta^2 & = & x^{-1} + x^{-2}\delta^2 \end{cases}.$$

This system is not feasible because x is an indeterminate and it is not equal neither 0 nor 1. However, if we take the obtained values for the coefficients and substitute in q and r then

$$q = x^{-1}\delta + \delta^2 \text{ and } r = 0,$$

and afterwards in Q and R we get

$$Q = x^{-3} + x^{-1}\delta + \delta^2 \text{ and } R = \delta - 1,$$

we see that indeed these two possible factors do not lead us to a factorization of the operator L.

- *For $s = 1$ we have we have $\widetilde{R} = x^{-1} + \delta$, $\widetilde{Q} = x^{-2}\delta$ and for q we get*

$$v_1(\widetilde{Q}) + 1 \leq v_1(x^j\delta^i) < v_1(\widetilde{Q}) + 2 \;\Rightarrow\; -2 \leq v_1(x^j\delta^i) < -1 \;\Rightarrow$$
$$v_1(x^j\delta^i) = -2,$$

for r we get

$$v_1(\widetilde{R}) + 1 \leq v_1(x^j\delta^i) < v_1(\widetilde{R}) + 2 \;\Rightarrow\; 0 \leq v_1(x^j\delta^i) < 1 \;\Rightarrow$$
$$v_1(x^j\delta^i) = 0,$$

in turn

$$q_{02} = 1 \text{ and } q_{j2} \text{ for } j \neq 0, \text{ and } r_{01} = 1 \text{ and } r_{j1} \text{ for } j \neq 0.$$

Whence,

$$r = r_{00} \text{ and } q = q_{-20}x^{-2} + q_{-11}x^{-1}\delta + \delta^2.$$

Substituting the product $\widetilde{Q}\widetilde{R}$ *and the operator* L *in the equality*

$$\widetilde{Q}\widetilde{R} + q\widetilde{R} + \widetilde{Q}r = L,$$

we obtain,

$$x^{-2}\delta(x^{-1} + \delta) + \widetilde{Q}r + q\widetilde{R} =$$
$$-x^{-3} - x^{-2} + x^{-1} + x^{-3}\delta + x^{-2}\delta - 2x^{-1}\delta + x^{-2}\delta^2 + 2x^{-1}\delta^2 + \delta^3.$$

After some computations and simplifications we get

$$q_{-20}x^{-3} + (q_{-20} + q_{-11} + r_{00})x^{-2}\delta - q_{-11}x^{-2} + q_{-11}x^{-1}\delta^2 =$$
$$-x^{-2} + x^{-2}\delta + x^{-1}\delta^2 \Rightarrow$$

$$\begin{cases} q_{-2,0} & = & 0 \\ q_{-2,0} + q_{-1,1} + r_{00} & = & 1 \\ q_{-1,1} & = & 1 \end{cases} \Rightarrow \begin{cases} q_{-2,0} & = & 0 \\ q_{-1,1} & = & 1 \\ r_{0,0} & = & 0 \end{cases}.$$

So,

$$q = x^{-1}\delta + \delta^2 \text{ and } r = 0.$$

Therefore,

$$Q = x^{-2}\delta + x^{-1}\delta + \delta^2 \text{ and } R = x^{-1} + \delta$$

are the factors of the only coprime 1 *factorization of the operator* L. *Actually,* Q *and* R *are a left-hand factor and a right-hand factor, respectively, of the operator* L.

4.4 The van Hoeij's Factorization Method for Computing a Right-Hand Factor

In this section we will present the van Hoiej's methods to factor differential operators that are not based on Beke's algorithm. In van Hoeij [1997b], he uses algorithms to find local factorizations (i.e. factors with coefficients in $\overline{k}((x))$, where k is a field of characteristic zero) and applies an adapted version of Padé approximation to produce a global factorization.

In order to do this, one should make a good choice of a singular point p of the operator L and a formal local right-hand factor of degree 1 at this point. After a translation of the variable ($x \mapsto x+p$ or $x \mapsto x^{-1}$) and a shift $\partial \mapsto \partial + e$ with $e \in \overline{k}(x)$, the operator L has a right-hand factor of the form $\partial - \frac{y'}{y}$ with an explicit $y \in \overline{k}[[x]]$. Now one tries to find out whether $\frac{y'}{y}$ belongs to $\overline{k}(x)$. Equivalently, one tries to find a linear relation between y and y' over $\overline{k}[x]$. This is carried out by a Padé approximation. The method extends to finding right-hand factors of higher degree and applies in that case a generalization of the Padé approximation.

This local-to-global approach has been implemented in Maple V.5. We start this section, which is extracted from van Hoeij [1997b], with some basic definitions like ramification of a field and ramification index, and the construction of the universal extension of a differential ring.

A **ramification** of the field $k((x))$ is a field extension $k((x)) \subset k((r))$, where r is algebraic over $k((x))$ with minimum polynomial $r^n - ax$ for a non-zero $a \in k$ and positive integer n. If $a = 1$ this is called a **pure ramification**.

For $r \in \overline{k((x))}$ (not necessarily with minimum polynomial $r^n - ax$), the **ramification index** of r, denoted by $\mathrm{ram}(r)$, is the smallest positive integer n for which $r \in \overline{k}((x^{1/n}))$. If K is a finite algebraic extension of $k((x))$ then the ramification index of K is the smallest positive integer n for which $K \subset \overline{k}((x^{1/n}))$.

All finite algebraic extensions $k((x)) \subset K$ are of the following form:

$$K = l((r))$$

where $k \subset l$ is a finite extension and $l((x)) \subset l((r))$ is a ramification.

Let V be the **universal extension** of $k((x))$, i.e., the differential ring extension of $\overline{k((x))}$ consistent of all solutions of all $L \in \mathcal{D} = k((x))[\delta]$, which is constructed as follows:

Define the set

$$E = \bigcup_{n \in \mathbb{N}} \overline{k}\left[x^{1/n}\right].$$

Consider $\mathrm{Exp}(e)$ and $\log(x)$ as indeterminates and define the free $\overline{k((x))}$-algebra W in these indeterminates

$$W = \overline{k((x))}\ [\{\,\mathrm{Exp}(e)\,|\,e \in E\,\}, \log(x)\,].$$

Define the derivatives

$$\mathrm{Exp}(e)' = \frac{e}{x}\mathrm{Exp}(e) \text{ and } \log(x)' = \frac{1}{x}.$$

This turns W into a differential ring. We can think of $\mathrm{Exp}(e)$ as

$$\mathrm{Exp}(e) = \exp\left(\int \frac{e}{x}\right)$$

because $x\frac{d}{dx}$ acts on $\mathrm{Exp}(e)$ as multiplication by e. Define the ideal I is generated by the following relations:

$$\mathrm{Exp}(e_1 + e_2) = \mathrm{Exp}(e_1) \cdot \mathrm{Exp}(e_2) \text{ for } e_1, e_2 \in E$$

and

$$\mathrm{Exp}(q) = x^q \in \overline{k((x))} \text{ for } q \in \mathbb{Q}.$$

This ideal is closed under differentiation. Now define V as the quotient ring $V = W/I$, hence V is a differential ring with $\overline{k}$ the field of constants of V. For $e \in E$ denote

$$V_e = \mathrm{Exp}(e).\left(\overline{k}.k((x))[e]\right)[\log(x)] \subset V.$$

Note that

$$\overline{k}.k((x))[e] = \overline{k}.k\left(\left(x^{1/n}\right)\right)[e]$$

where n is the ramification index of e. Define $\sim$ on E as follows:

$$e_1 \sim e_2 \ \Leftrightarrow\ e_1 - e_2 \in \frac{1}{\mathrm{ram}(e_1)}\mathbb{Z}$$

i.e., $e_1 - e_2$ is an integer divided by the ramification index of e_1. Then

$$V_{e_1} = V_{e_2} \ \Leftrightarrow\ e_1 \sim e_2$$

so V_e is defined for $e \in E/\sim$. Denote the set of solutions of $L \in \mathcal{D}\backslash\{0\}$ in V as $V(L)$. This is a $\overline{k}$-vector space. Since every $L \in \mathcal{D}\backslash\{0\}$ has a fundamental set of solutions in V it follows that

$$\dim(V(L)) = \mathrm{ord}(L).$$

The number $\dim(V(L))$ is useful for factorization because it is independent of the order of multiplication, i.e.

$$\dim(V(fg)) = \dim(V(gf)).$$

Now split $V(L)$ in a direct sum and look at the dimension of the components

$$V = \bigoplus_{e \in E/\sim} V_e.$$

The V_e are $\overline{k}$-linear spaces and also $\mathcal{D}$-modules. So

$$V_e(L) \subset V_e \text{ for all } L \in \mathcal{D}\backslash\{0\}$$

Then $L(V_e) = V_e$ because L is surjective on V. The kernel of L on V_e is denoted by

$$V_e(L) = V(L) \cap V_e.$$

Denote

$$\mu_e(L) = \dim\left(V_e(L)\right).$$

This number is called the **multiplicity** of e in L. The multiplicities μ_e are useful for factorization because they are also independent of the order of the multiplication, e.i., if $f, g \in \mathcal{D}\backslash\{0\}$ then

$$\mu_e(gf) = \mu_e(fg) = \mu_e(f) + \mu(g).$$

It is also follows from the fact that the dimension of the kernel of the composition of two surjective maps equals the sum of the dimension of the kernels.

An element $e \in E/\sim$ is called an **exponential part** of L if $\mu_e(L) > 0$. The sum of the multiplicities of all exponential parts of L equals the order of L.

Let $e \in \overline{k((x))}$. Then the **substitution map** $S_e : \mathcal{D} \to \mathcal{D}$ is a $k((x))$-homomorphism defined by

$$S_e(\delta) = \delta + e,$$

which is a ring automorphism. Then,

$$V(L) = \text{Exp}(e) \cdot V(S_e(L)).$$

Let $L \in \mathcal{D}\backslash\{0\}$ and $e \in E$. Let n be the ramification index of e. Let $P = N_0\left(S_e(L)\right)$ be the Newton polynomial corresponding to slope 0 in the Newton polygon of $S_e(L) \in \overline{k}((x^{1/n}))[\delta]$. Now $\mu_e(L)$ is defined as the number of roots (counted with multiplicity) of P in $\frac{1}{n}\mathbb{Z}$. If $e_1 \sim e_2$ then

$$\mu_{e_1}(L) = \mu_{e_2}(L) \text{ for all } L \in \mathcal{D}\backslash\{0\}$$

hence $\mu_e(L)$ is defined for $e \in E/\sim$ as well.

Let K be a finite algebraic extension of $k((x))$ and let $L \in K[\delta]$. Then L is called **semi-regular** over K if L has a fundamental system of solutions in $K[\log(x)]$. This is equivalent to the following two conditions

- L is regular singular.
- The roots of the Newton polynomial $N_0(L)$ are integers divided by the ramification index of K over $k((x))$.

Note that the definition of semi-regular depends on the field K. For $L \in \mathcal{D}$ we have

$$\mu_0(L) = \text{ord}(L)$$

if and only if all solutions of L are elements of

$$V_0 = \overline{k} \cdot k((x))[\log(x)]$$

if and only if L is semi-regular over $k((x))$. A regular operator is semi-regular as well.

For a point $p \in P^1(\overline{k}) = \overline{k} \cup \{\infty\}$ let $l_p : \overline{k} \to \overline{k}$ be the $\overline{k}$-automorphism defined by

$$l_p(x) = \begin{cases} x+p & \text{if } p \in \overline{k} \\ \\ 1/x & \text{if } p = \infty \end{cases}.$$

It can be extended to a ring automorphism of $\overline{k}(x)[\partial]$ by defining

$$l_p(\partial) = \begin{cases} \partial & \text{if } p \in \overline{k} \\ \\ -x^2\partial & \text{if } p = \infty \end{cases}.$$

For any differential operator $L \in \overline{k}(x)[\partial]$ the operator $l_p(L)$ viewed as an element of $\overline{k}((x))[\delta]$ instead of $\overline{k}((x))[\partial]$, is called the **localization of L at the point** $x = p$.

Let $e \in E/\sim$, $L \in k(x)[\partial]$ and $p \in P^1(\overline{k})$. Define

$$\mu_{e,p}(L) = \mu_e(l_p(L)).$$

If $\mu_{e,p}(L) > 0$ then e is called an **exponential part of L at the point** p, and the number $\mu_{e,p}(L)$ is called the **multiplicity of e at the point** p.

If p is a semi-regular point of L then L has only a trivial (i.e., zero module $\sim$) exponential part at p.

Denote by $\mu_*(L) : (E/\sim) \times P^1(\overline{k}) \to \mathbb{N}$ the function which maps (e,p) to $\mu_{e,p}(L)$. Then, for $f, g \in \overline{k}(x)[\partial]$ we have

$$\mu_*(fg) = \mu_*(f) + \mu_*(g).$$

Let $L \in k(x)[\partial]$ and suppose a non-trivial factorization $L = QR$ exists with $Q, R \in \overline{k}(x)[\partial]$. We want to determine a right-factor of L. This could be done if we knew a non-zero subspace $W \subset V(R)$. We only know that $V(R) \subset V(L)$ but this does not give any non-zero element of $V(R)$.

For any exponential part e of L at a point $p \in P^1(\overline{k})$ we have (after replacing f, L, R by $l_p(L), l_p(Q), l_p(R)$ we may assume that $p = 0$)

$$V_e(R) \subset V_e(L) \text{ and } \mu_e(Q) + \mu_e(R) = \mu_e(L).$$

Suppose that we are in a situation where $\mu_e(Q) = 0$. Then the dimension of $V_e(R)$ and $V_e(L)$ are the same and hence we have found a subspace $V_e(L) = V_e(R)$ of $V(R)$, i.e.,

$$V_e(L) \subset V(R).$$

Then we can factor L. Note that we do not necessarily find the factorization QR, it is possible that instead of R a right-hand factor of R is found. In other words

$$S_e(R_e) \in k((x))[e, \delta]$$

is a right-hand factor of R, where R_e is the semi-regular part of f.

We want to have a local right-hand factor r of R. There are several strategies:

1. We can take $r = S_{-e}(R_e)$, or we can take a first order right-hand factor in $k((x))[e, \delta]$ of $S_{-e}(R_e)$.
2. Another strategy, to speed up the algorithm, is first to try to factor L in $k(x)[\partial]$ instead of $\overline{k}(x)[\partial]$. If no factorization in $k(x)[\partial]$ is obtained, then we can get rid of the computations afterwards to search a factorization in $\overline{k}(x)[\partial]$.

If we want to factor L in $k(x)[\partial]$ then we can take $r \in \mathcal{D}$ of minimal order such that $S_{-e}(R_e)$ is a right-hand factor of r. So, depending on whether we want to factor L in $k(x)[\partial]$ or in $\overline{k}(x)[\partial]$, we have a right-factor $r \in k((x))[\delta]$ or $r \in k((x))[e, \delta]$ of R.

From now on assume that $r \in \mathcal{D}$, the other case works precisely the same (just replace k by $\overline{k}$).

Let $n = \mathrm{ord}(L)$. The goal is to compute an operator

$$R = a_d \partial^d + \cdots + a_0 \partial^0 \in k[x, \partial]$$

that has r as a right-hand factor. Here d should be minimal. Because r divides both L and R on the right it also divides $\mathrm{gcrd}(L, R)$. Then

$$\mathrm{gcrd}(L, R) = R$$

because d is minimal. We conclude that R is a right-hand factor of L. If $d < n$ a non-trivial factorization is obtained this way.

There are two ways to choosing the number d. The first is to try all values $d = 1, 2, \ldots, n-1$. Suppose that for a certain d we find an R that has r as a right-hand factor and for numbers smaller than d such R could not found. Then d is minimal and hence R is a right-hand factor of L.

The second approach to take $d = n - 1$. If we find

$$R = a_d \partial^d + \cdots + a_0 \partial^0 \in k[x, \partial]$$

that has r as a right-factor we can compute $\mathrm{gcrd}(L, R)$. This way we also find a right-factor of L.

Sometimes it is possible to conclude a priori that there is no right-hand factor of order $n-1$. If for instance all irreducible local factors have order ≥ 3 then the order of a right-hand factor if ≤ 3 and so we can take $d = n - 3$ instead $d = n - 1$.

We can compute a bound N for the degree of the a_i. So the problem now is:

Are there polynomials $a_i \in k[x]$ of degree $\leq N$, not all equal to 0, such that r is a right-hand factor of

$$R = a_d \partial^d + \cdots + a_0 \partial^0 \ ?$$

Let m be the order of r. The $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}r$ is a $k((x))$-vector space of dimension m with basis $\partial^0, \partial^1, \ldots, \partial^{m-1}$. Write $\partial^0, \partial^1, \ldots, \partial^d$ on this basis as vectors $v_0, \ldots, v_d$ in $k((x))^m$. Now multiply $v_0, \ldots, v_d$ with a suitable power of x such that the v_i become elements of $k[[x]]^m$. r is a right-factor of R if and only if

$$a_0 v_0 + \cdots + a_d v_d = 0$$

in $k[[x]]^m$. This is a system of linear equations with coefficients in $k[[x]]$ which should be solved over $k[x]$. One way of solving this is to convert it to a system of linear equations over k using the bound N. A much faster way is the Beckermann-Labahn algorithm Beckermann and Labahn [1994]. Their method is the following.

Sketch of the Beckermann-Labahn algorithm

- Let $M_i \subset k[x]^{d+1}$ be the $k[x]$-module of all sequences $(a_0, a_1, \ldots, a_d)$ for which

  $$v(a_0 v_0 + \cdots + a_d v_d) \geq i.$$

  The "valuation" v of a vector is defined as the minimum of the valuation of its entries. The valuation of 0 is infinity.

- Choose a basis (as $k[x]$-module) of M_0.
- For $i = 1, 2, 3, \ldots$ compute a basis of M_i using the basis for M_{i-1}.

Define the degree of a vector of polynomials as the maximum of the degree of these polynomials. From the basis for M_i we can find a non-zero polynomial $A_i \in M_i$ with minimal degree. Suppose there exists such

$$R = a_d\partial^d + \cdots + a_0\partial^0 \in k[x, \partial]$$

having r as a right-hand factor. Then there exists such R with all

$$\deg(a_i) \leq N$$

where N is a bound we can compute. So then there is a non-zero $(a_0, \ldots, a_d)$ of degree $\leq N$ which is an element of every M_i. Because of the minimality of $\deg(A_i)$ it follows that then $\deg(A_i) \leq N$ for all i. So whenever $\deg(A_i) > N$ for any i we know that there is no $R \in k(x)[\partial]$ of order d which has r as a right-hand factor.

Algorithm Construct R

For $i = 0, 1, 2, \ldots$ do

- Compute M_i and $A_i \in M_i$ of minimal degree.
- If $\deg(A_i) > N$ then RETURN "R does not exist".
- If $\deg(A_i) = \deg(A_{i-3})$ then

  Comment: The degree did not increase 3 steps in a row so it is likely that a right-hand factor is found.

  If $A_i = (a_0, \ldots, a_d)$ then write $R = a_d\partial^d + \cdots + a_0\partial^0$. Divide by a_d to make R monic. Test if R and f have a non-trivial right-hand factor in common. If so, return this right-hand factor, otherwise continue with the next i.

5. CONCLUSIONS

In this thesis, starting from an overview of symbolic factorization of linear differential operators, a new result – a procedure for factoring second order partial differential operators – was presented. Beginning with the ring of linear differential operators the research took us – through differential Galois theory and the body of methods for finding rational and exponential solutions of linear differential equations – to the state of the art: Singer's eigenring factorization algorithm, factorization via Newton's polygons, and the van Hoeij's methods for local factorization.

In Chapter 1, we presented an algorithmic solution of the problem of factoring second order linear partial differential operators. Based on purely algebraic methods and by using techniques of differential algebra, we solved the problem in its original settings.

In contrast with the case of ordinary differential operators, the advantage of this approach is that, in the most general case, it is not necessary to solve a Riccati equation for partial differential operators.

Comparing our method with other known ones, we have found that:

- Compared with (Miller [1932]), our work is characterized by the following.
  - We do not only propose a possible right factor but rather we find the factorization when it exists, without appealing to the necessity to define new structures or to extend the original domain in which we are working.
  - While Miller found only a possible right-hand factor, we devise a complete factorization of the operator, and, in each case, we avoid dividing the operator by the right-hand factor.
  - We do not make a case distinction, since we propose to find a square root and to solve a system of two linear equations in two unknowns plus one first order linear partial differential equation.
  - If the square root exists and if the system of algebraic equations which we obtain has a unique solution, we consider the extra linear partial differential equation. The extra partial differential equation is used as a test equation, rather than been used as an additional condition.

- Comparing our approach with the Hensel descent of Grigoriev and Schwarz [2004], for second order linear differential operators we have found following.
  - We need neither to define new objects nor to work in another algebraic structure, because we find the factorization at once in the domain of definition of the linear differential operators.
  - Instead of solving the problem by reformulating it in a commutative ring, we solve it in the original non-commutative setting.
  - We do not need to apply undetermined coefficients, after applying polynomial factorization in each particular case. This reduces the complexity of our approach.

We would like to mention also that:

- The structure of Theorem (2) proposes an algorithmic method, **algorithm** $gl1$, whose proof of correctness is given by the proof of Theorem (2). This method can be easily implemented in any computer algebra system;

- The method above has been generalized to operators of third order, but this results in a more complicated system of equations. In contrast to the second order case, we must solve certain differential equations, which, in particular cases, are simplified with the aid of characteristic sets.

In Chapter 2 we have introduced a formula for the complete factorization of a given element of the ring of linear differential operators. This factorization is always possible when a fundamental set of solutions of a differential equation is available. With the formula above we have explained, in a direct way, the classical Beke algorithm (Beke [1894]) and its variants, as well as the algorithm LODEF by Schwarz [1989], and the Beke-Bronstein algorithm Bronstein [1994].

We can summarize Beke's algorithm in the following way:

- For first order right-hand factors, the idea is to find a rational solution of the Riccati equation associated with the linear scalar equation defined by the linear differential operator. If this Riccati equation has a rational solution, then it is also at once an exponential solution of the linear scalar equation. This exponential solution yields a first order right-hand factor.
- For searching right-hand factors of higher order, the idea is to determine the coefficients of the possible right-hand factor of order m, where $m < n$ and n is the order of the given operator. After having chosen m we have to express the associated equations for it (i.e., the differential equations in the generalized Wronskians, of a subset of m elements, of a fundamental set of solutions of the scalar equation); solve the equations for rational solutions; and construct a right-hand factor. If one of the associated equations has not rational solution, then stop the process by saying that there is not a right-hand factor of order m. Then try another m, and so on.

After almost one hundred years, Schwarz [1989] automated Beke's algorithm, extending it to search recursively from possible right-hand factors of order $n-1$, until possible right-hand factors of order 1. He described also a different way to find the associated equations, and implemented this procedure – the algorithm LODEF – in the Scrachpad II computer algebra system. He then analysised the cost of factorization of linear differential operators with rational function coefficients, estimating bounds for the size of the polynomials in the numerator of the rational function solutions. With these ideas, he developed the RiccatiRational algorithm, to complete the last step of the Beke algorithm, namely to solve the generalized Riccati equations derived from the associated equations.

A faster approach – the Beke-Bronstein algorithm, Bronstein [1994] – is an efficient procedure for computing the associated equations, based on elementary operations on sets of positive integers. It produces many possible choices of the associated equation. The algorithm is designed to select the simplest equations for solving. These ideas were formalized, Two years later in Bronstein and Petkovšek [1996], in the so called "Pseudo-Linear Algebra". The resulting theory was formulated afterwards as operations in differential modules (Chapter 2 of van der Put and Singer [2003]).

There are several algorithms for finding rational solutions of linear differential equations, some of them depending on the coefficient field and others on the order of the equation. For our explanation of how to find rational solutions of linear differential equations, we have chosen an algorithm for finding solutions of differential equations with rational function coefficients. The procedure depends on a theorem which says that a solution can only have a pole at either α or ∞, if at least one of the coefficients has pole either at α or ∞. After identifying the poles of the coefficients, we expand them in Laurent series at each of the poles, we use indicial equations to find bounds on the numerator of the coefficients of the solution, and with linear algebra we can find the numerators and hence the solution.

In contrast to the case of rational solutions of linear differential equations, there are few algorithms for finding either exponential solutions of linear differential equation or rational solutions of the associated Riccati equation, only for particular cases. We have discussed the RiccatiRational algorithm – Schwarz [1994] – for finding rational solutions in the coefficient field of the Riccati equation associated with a given linear differential equation. The algorithm searches for bounds on the coefficients of the possible solution and is ultimately reduced to solving a system of linear equations. If the system is feasible, we obtain a rational solution of the associated Riccati equation and at once a right-hand factor of the original linear differential equation. In our example, we have simplified the application of the algorithm, namely the computation of the bounds given in Schwarz [1989], using the techniques commonly applied only to rational solutions in the general case.

Recently Cluzeau and van Hoeij [2004] presented a new algorithm, based on local and modular computation, for finding exponential solutions of differential equations with rational function coefficients. The approach reduces the number of possibilities in the combinatorial part of the algorithm. The authors have also showed how unnecessary extensions of the constants can be avoided. Their idea is to use information $\mod p$ in several ways, in order to speed up the computations the exponential solutions in characteristic 0. They also devised a recursive algorithm – Algorithm FindASol – for constructing a field extension over which an exponential solution, if one exists, can be found.

We concluded this work, in Chapter 3, by presenting the known algorithms for commutative polynomials with a view to adaptation, generalization, and reformulation of their analogues for linear differential operators, as follows

- Berlekamp algorithm for the Singer's eigenring algorithm (Singer [1996]),
- Newton polygons for geometric factorization and module decomposition (van der Put and Singer [2003]),
- Puiseux series and local parametrization of curves for factorization via Newton polygons and the coprime index 1 (van Hoeij [1997a]),
- Padé approximation for the van Hoeij's algorithm for finding right-hand factors (van Hoeij [1997b]).

In conclusion, this work presents the results of our studies in differential Galois theory, as applied to symbolic factorization of differential operators. Starting from the clarification and systematization of the theoretical foundations relevant to factorization, we have evolved original ideas about the factorization of partial diffential operators and provided the thoretical foundations and algorithmic specifications for second order linear operators. Furthemore, we have complemented the theoretical work with a number of concrete examples chosen from relevant application areas.

This work can be extended in a number of ways. On the theoretical side, we feel that interesting future work could be in the area of differential elimination, where our results may be instrumental in solving linear differential equations by operators. On the applications side, the our results can complement a computer-algebra system, by proving tools for solving a class of partial differential equations, by factorization.

6. LIST OF NOTATION

- Chapter 2
  - $\mathcal{R}$ ring, k field, 10
  - $\delta : \mathcal{R} \to \mathcal{R}$ derivation on the ring $\mathcal{R}$, 10
  - $(\mathcal{R}, \delta)$ (reps. (k, δ)) differential ring (reps. field), 10
  - $Const_\delta(\mathcal{R})$ subring of constants of $\mathcal{R}$ with respect to δ, 10
  - $\mathcal{S}$ differential subring (resp. subfield) of $\mathcal{R}$, 10
  - $\mathcal{R}\{\{y_1, \ldots, y_n\}\}$ ring of differential polynomials in the inderminates $y_1, \ldots, y_n$ over $\mathcal{R}$, 10
  - $k[\partial_1, \ldots, \partial_r]$ ring of partial differential operators with coefficients in k, 11
  - $\mathcal{D} := k[\partial]$ ring of linear ordinary differential operators with coefficients in k, 12
  - L linear differential operator, 12
  - $\deg(L) = \operatorname{ord}(L)$ degree or order of L, 12
  - $s(L)$ symbol of L, 21
- Chapter 3
  - A_L companion matrix of L, 25
  - $W(y_1, y_2, \ldots, y_n)$ Wronskian matrix of $y_1, y_2, \ldots, y_n \in k$, 26
  - $w(y_1, y_2, \ldots, y_n)$ Wronskian of $y_1, y_2, \ldots, y_n \in k$, 26
  - $G(K/k)$ differential Galois group of the extension K of k, 27
  - $Gal(K/k)$ Galois group of a differential equation $L(y) = 0$ with coefficients in k and Picard-Vessiot extension K, 28
  - $\overline{K}$ algebraic closure of a field K, 33
  - $k(x)$ field of rational functions over k, 33
  - $k[[x]]$ ring of formal power series in the indeterminate x with coefficients in k, 33
  - $k((x))$ field of formal Laurent series in the x with finite pole order and coefficients in k, 33
  - $\operatorname{ord}(a)$ order of $a \in k((x))$, 33
  - $I_p(m)$ indicial polynomial of $p \in k((x))$, 34
  - $ord_s(f)$ order of $f \in k((x))$ at s, 35
  - $S = \{s_i | 1 \leq s_1 < \cdots < s_m \leq n\}$, 51
  - A_S square submatrix of $A \in k^{n \times m}$, 52
  - S_i^+ increment the i-th element of S, 53
  - $S_i^{[l]}$ replace the i-th element of S by l, 53
  - $\delta_i^{[l]}(S)$ number of elements of S which are strictly between l and s_i, 53
- Chapter 4
  - $\mathcal{E}_{\mathcal{D}}(L)$ eigenring of L, 64, 66, 67
  - $\operatorname{gcrd}(L_1, L_2)$ greatest common right divisor of L_1 and L_2, 67
  - $[L_1, L_2]_l$ least common left multiple of L_1 and L_2, 67
  - $N(L)$ Newton polygon of L, 70

7. LIST OF ALGORITHMS

8. LIST OF FIGURES

INDEX

BIBLIOGRAPHY

Ian M. Anderson and Niky Kamran. The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane. *Duke Mathematical Journal*, 87(2):265–319, 1997.

B. Beckermann and G. Labahn. A uniform approach for the fast computation of Matrix-type Padé approximants. *SIAM J. Matrix Analysis and Applications*, 16:804–823, 1994.

Emanuel Beke. Die Irreducibilität der homogenen linearen Differentialgleichungen. *Mathematische Annalen*, 45:185–195, 1894.

Henry Blumberg. *Über algebraishe Eigenschaften von linearen homogenen Differentialausdrücken*. PhD thesis, Göttingen, 1912. Inaugural-Dissertation.

Manuel Bronstein. *Symbolic Integration I - Trascendental Functions*, volume 1 of *Algorithms and Computation in Mathematics*. Springer, Berlin, Heidelberg, New York, Barcelona, Budapest, Hong Kong, London, Milan, Paris, Santa Clara, Singapore, Tokio, 1997.

Manuel Bronstein. Linear differential equations: Breaking through the order 2 barrier. In Paul S. Wang, editor, *Proceedings of the 1992 International Symposium on Symbolic and Algebraic Computation ISSAC'92, Berkeley, California, USA*, pages 42–48. ACM Press, 1992a. Available online at the URL http://www-sop.inria.fr/cafe/Manuel.Bronstein/publications/mb_papers.html.

Manuel Bronstein. On solutions of linear differential equations in their coefficient field. *Journal of Symbolic Computation*, 13:413–439, 1992b. Available online at the URL http://www-sop.inria.fr/cafe/Manuel.Bronstein/publications/mb_papers.html.

Manuel Bronstein. An improved algorithm for factoring linear ordinary differential operators. In Joachim von zur Gathen, editor, *Proceedings of the 1994 International Symposium on Symbolic and Algebraic Computation ISSAC'94, Oxford, England, United Kingdom*, pages 336–340. ACM Press, 1994.

Manuel Bronstein and Marko Petkovšek. An introduction to pseudo-linear algebra. *Theoretical Computer Science*, 157(1):3–33, 1996. Available online at the URL http://www-sop.inria.fr/cafe/Manuel.Bronstein/publications/mb_papers.html, and http://citeseer.nj.nec.com/bronstein96introduction.html.

Eduardo Casas-Alvero. *Singularities of Plane Curves*. Number 276 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2000.

Thomas Cluzeau and Mark van Hoeij. A modular algorithm for computing the exponential solutions of a linear differential operator. *Journal Symbolic Computation*, 38(3):1043–1076, 2004.

Arjeh M. Cohen, Hans Cuypers, and Hans Sterk, editors. *Some Tapas in Computer Algebra*, volume 4 of *Algorithms and Computation in Mathematics*. Springer, Berlin, Heidelberg, New York, Barcelona, Hong Kong, London, Milan, Paris, Singapore, Tokyo, 1999.

Jean Gaston Darboux. *Ann. École Normale Supérieure*, 7:163–173, 1870.

Winfried Fakler. On second order homogeneous linear differential equations with Liouvillian solutions. *Theoretical Computer Science*, 187(1-2):27–48, 1997. Available online at the URL http://www.citeseer.nj.ne.com/fakler97second.html.

L. Fuchs. Über die linearen Differentialgleichungen zweiter Ordnung, welche algebraische Integrale besitzen, und eine neue Anwendung der Invariantentheory. *Journal für die reine und angewandte Mathematik*, 81, 1875.

L. Fuchs. Über die linearen Differentialgleichungen zweiter Ordnung, welche algebraische Integrale besitzen, zweite Abhandlung. *Journal für die reine und angewandte Mathematik*, 85, 1878.

M. Griesbrecht. Factoring in skew-polynomial rings. Preprint, 1992. Department of Computer Science, University of Toronto.

D. Yu Grigoriev. Complexity of factoring and calculating the gcd of linear ordinary differential operators. *Journal of Symbolic Computation*, 10:7–37, 1990a.

D. Yu Grigoriev. Complexity of irreducibility testing for a system of linear ordinary differential equations. In Shunro Watanabe and Morio Nagata, editors, *Proceedings of the 1990 International Symposium on Symbolic and Algebraic Computation ISSAC'90, Tokyo, Japan*, pages 225–230. ACM Press, 1990b.

Dima Grigoriev and Fritz Schwarz. Factoring and solving linear partial differential equations. *Computing*, 73:179–197, 2004.

E. L. Ince. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

C. Jordan. Mémorie sur les équations différentielles linéaires á integrale algébrique. *Journal für die reine und angewandte Mathematik*, 84, 1878.

M. Juras. Generalized Laplace invariants and classical integration methods for second-order scalar hyperbolic partial differential equations in the plane. In J. Janyske, editor, *Proceedings of the 6th International Conference on "Differential Geometry and Applications", Masaryk Univ., Brno, Czech Republik, August 28 - September 1*, pages 275–284, 1995.

Irving Kaplansky. *Introduction to Differential Algebra*. Actualités Scientifiques et Industrielles 1251. Publications de L'Institut de Mathématique de L'Université de Nancago. Hermann, 6, Rue de la Sorbonne, Paris V, 1957.

E. R. Kolchin. Algebraic matrix groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations. *Annals of Mathematics*, 49, 1948.

Jerald J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. *Journal of Symbolic Computation*, 2:3–43, 1986.

Edmund Landau. Ein satz über die zerlegung homogener linearer Differentialausdrücke in irreducible Factoren. *Journal fü die reine und angewandte Mathematik*, 124:115–120, 1902.

Ziming Li and Fritz Schwarz. Rational solutions of riccati-like partial differential equations. *Journal of Symbolic Computation*, 31:691–716, 2001.

Ziming Li, Fritz Schwarz, and Serguei P. Tsarev. Factoring systems of linear pdes with finite-dimensional solution spaces. *Journal of Symbolic Computation*, 36:443–471, 2003.

Joseph Liouville. Premier mémoire sur la détermination des intégrales dont la valeur est algébrique. *Journal de l'École polytechnique*, 14:124–148, 1833a.

Joseph Liouville. Second mémoire sur la détermination des intégrales dont la valeur est algébrique. *Journal de l'École polytechnique*, 14:149–193, 1833b.

A. Loewy. Überber vollständig reduzible lineare homogene Differentialgleichungen. *Mathematische Annalen*, 62:89–117, 1906.

B. Malgrange. Sur la réduction formelle des équations différentielles à singularités irrégulières, 1979. Manuscript.

F. H. Miller. *Reducible and irreducible linear differential operators*. PhD thesis, Columbia University, 1932.

P. Th. Pépin. Méthode pour obtenir les intégrales algébriques des équationes différentielles linéaires du second ordre. *ATTi dell' Accad. Pont. de Nuovi Lincei*, XXXIV:243–388, 1881.

J.-T. Ramis. Dévissage gevrey. *Astérisque*, 59/60:173–204, 1978.

Fritz Schwarz. A factorization algorithm for linear ordinary differential equations. In Gaston H. Gonnet, editor, *Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation ISSAC'89, Portland, Oregon, USA*, pages 17–25. ACM Press, 1989.

Fritz Schwarz. Efficient factorization of linear ode's. *A Quartely Publication of the SIGSAM Bolletin. Special Interest Group on Symbolic and Algebraic Manipulation*, 28(1):9–17, 1994.

Michael F. Singer. Algebraic solutions of linear differential equations. In *Proceedings of the Queen's University 1979 Conference on Number Theory*, volume 62(2), pages 379–420, 1979.

Michael F. Singer. Liouvillian solutions of nth homogeneous linear differential equations. *American Journal of Mathematics*, 103(4):661–682, 1981.

Michael F. Singer. An outline of differential Galois theory. In E. Tournier, editor, *Computer Algebra and Differential Equations*, pages 3–58, London, San Diego, New York, Berkeley, Boston, Sydney, Tokyo, Toronto, 1989. Academic Press.

Michael F. Singer. Liouvillian solutions of linear differential equations with Liouvillian coefficients. *Journal of Symbolic Computation*, 11(3):251–274, 1991.

Michael F. Singer. Testing reducibility of linear differential operators: A group theory perspective. *Applicable Algebra in Engineering, Communication and Computing*, 7(2):77–104, 1996. Avalaible on line at the URL http://www4.ncsu.edu/ singer/ms_papers.html.

Michael F. Singer and Felix Ulmer. Liouvillian and algebraic solutions of second and third order linear homogeneous differential equations. *Journal of Symbolic Computation*, 16:37–73, 1993. Available online at the URL http://www4.ncsu.edu/ singer/ms_papers.html.

S. P. Tsarev. An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator. In Y. N. Lakshman, editor, *Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation ISSAC'96, Zurich, Switzerland*, pages 226–231. ACM Press, 1996.

S. P. Tsarev. Factoring linear partial differential operators and the Darboux method for integrating nonlinear partial differential equations. *Theoretical and Mathematical Physics*, 122(1):121–133, 2000.

Marius van der Put and Michael F. Singer. *Galois Theory of Linear Differential Equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], A Series of Comprehensive Studies in Mathematics*. Springer-Verlag, Berlin, 2003. Available online at the URL http://www4.ncsu.edu/ singer/ms_papers.html.

Mark van Hoeij. Formal solutions and factorization of differential operators with power series coefficients. *Journal of Symbolic Computation*, 24:1–30, 1997a. Available online at the URL http://www.math.fsu.edu/ hoeij/papers.html.

Mark van Hoeij. Factorization of differential operators with rational functions coefficients. *Journal of Symbolic Computation*, 24:537–561, 1997b. Available online at the URL http://www.math.fsu.edu/ hoeij/papers.html.

Jacques-Arthur Weil. The use of the special semi-group for solving differential equations. In Joachim von zur Gathen, editor, *Proceedings of the 1994 International Symposium on Symbolic and Algebraic Computation ISSAC'94, Oxford, England, United Kingdom*, pages 341–347. ACM Press, 1994.

A. V. Zhiber, V. V. Sokolov, and S. Ya Startsev. On nonlinear Darboux-integrable hyperbolic equations. *Doklady RAN*, 343(6):746–748, 1995. In Russian.

CURRICULUM VITAE

Glauco Alfredo López Díaz

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University, Linz, Austria

Tel: +43 (0)732 2468 9955 Fax: +43 (0)732 2468 9930
E-mail: gdiaz@risc.uni-linz.ac.at
Homepage: http://www.risc.uni-linz.ac.at/people/gdiaz

Personal Data

Family name:	López Díaz
First name:	Glauco Alfredo
Date of birth:	4th June, 1963
Place of birth:	Caracas, D.F. Venezuela
Citizenship:	Venezuelan
Private address:	Sankt Peter Straße 31, A-4240 Freistadt, Austria

Education

2001 – : Currently, I am a Ph.D. Student of Technical Sciences.
Research Institute for Symbolic Computation (RISC-Linz),
Johannes Kepler University, Linz, Austria.
Research topic: Symbolic Differential Computation.

1991 – 1998: M.Sc. in Mathematics.
"Univeridad de Los Andes" (University of The Andes), ULA-Mérida,
Mérida, Venezuela.
Master Thesis: Bernoulli Polynomials and Bernoulli Numbers.
Supervisor: Prof. Francisco Rivero Ph.D..

1982 – 1989: Bachelor in Mathematics.
"Universidad Central de Venezuela" (Central University of Venezuela), UCV,
Caracas, Venezuela.
Bachelor Thesis: On the set of the non-wandering points over a compact
two-dimensional manifold.
Supervisor: Prof. Pedro Echeverria Ph.D..
Winner of a Special Graduation Award.

1976 – 1981: High School "Colegio Fe y Alegría", La Rinconada, Caracas, Venezuela.

Research Interests

- Symbolic Computation
- Computer Algebra
- Differential Galois Theory
- Differential Elimination
- Symbolic Differential Computation

Technical Reports

Rahul Ramesh Athale, Glauco Alfredo López Díaz.
Explaining Schwarz's lodef algorithm with examples
Technical report no. 03-11 in RISC-Linz Report Series, RISC-Linz, Johannes Kepler University, Linz, Austria, November 2003.

Glauco Alfredo López Díaz.
Una nueva fórmula de recurrencia para los números de Bernoulli
Technical report no. 213 in Notas de Matemáticas, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela, 2001.

Glauco Alfredo López Díaz.
Propiedades importantes de los números de Bernoulli
Technical report no. 212 in Notas de Matemáticas, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela, 2001.

Glauco Alfredo López Díaz.
Polinomios de Bernoulli y números de Bernoulli
Technical report no. 211 in Notas de Matemáticas, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela, 2001.

Professional Experience

Since September 2004, I am employed by the project DET, Fonds zur Förderung der wiss. Forschung, P16357-N04, at the Research Institute for Symbolic Computation (RISC-Linz), Johannes Kepler University, Linz, Austria.
Supervisor: O. Uni.Prof. Dipl.-Ing. Dr. Franz Winkler.

September 2003 - March 2004, I was employed by the project DET, Fonds zur Förderung der wiss. Forschung, P16357-N04, at the Research Institute for Symbolic Computation (RISC-Linz), Johannes Kepler University, Linz, Austria.
Supervisor: O. Uni.Prof. Dipl.-Ing. Dr. Franz Winkler.

Since October 2001, I am a Ph.D. student at RISC-Linz in Austria.

Since January 1999: Assistant Professor of the Department of Mathematics, College of Science, University of The Andes, Mérida, Venezuela.

Since February 1993: Ordinarious Member of the Department of Mathematics, College of Science, University of The Andes, Mérida, Venezuela.

May 1991 - December 1998: Instructor of the Department of Mathematics, College of Science, University of The Andes, Mérida, Venezuela.

March 1990 - April 1991: Lecturer at the College of Engineering, "Universidad José Maria Vargas" (José Maria Vargas University), Caracas, Venezuela.
Topics: Mathematics I, Mathematics II, Mathematics IV, Analitic Geometry.

March 1989 - March 1990: Lecturer at the "Instituto Universitario de Tecnologia Rodolfo Loero Arismendi" (Lodolfo Loero Arismendi University Institute of Technology), I.U.T.I.R.L.A., Caracas, Venezuela.
Topics: Mathematics I, Logic, Statistics II.

January 1989 - April 1991: Lecturer at the College of Engineering, "Instutito Universitario de Nuevas Profesiones" (New Professions University Institute), Caracas, Venezuela.
Topics: Mathematics I, Mathematics II, Mathematics III, Numerical Analysis.

Other Experiences

January - August 1988: Operator at the Service of Trains, "Metro de Caracas C.A.", Caracas, Venezuela.

March 1986 - April 1987: Trainer II for the practice lectures at the College of Science, Central University of Venezuela, Caracas, Venezuela.
Topics: Mathematics III, Mathematics IV.

October 1984 - March 1986: Trainer I for the practice lectures at the College of Science, Central Univeristy of Venezuela, Caracas, Venezuela.
Topics: Mathematics IV, Mathematics II, Mathematics III.

Special Courses and Workshops

September 1995: VII Venezuelan School of Mathematics, University of The Andes, Mérida, Venezuela. *Course*: Eliptic Curves in Criptography.

March - April 1993: VI Workshop in Mathematics, "Universidad Simon Bolivar" (Simon Bolivar University), Caracas, Venezuela.

September 1992: V Venezuelan School of Mathematics, University of The Andes, Mérida, Venezuela. *Course*: Computational Methods in the Theory of Operators over Hilbert Spaces.

January 1992: International Research Workshop on Banach Space Theory, University of The Andes, College of Scince, Department de Matemáticas, Mérida, Venezuela.

September 1991: IV Venezuelan School of Mathematics, University of The Andes, Mérida, Venezuela. *Course*: Multilinear Algebra and Free Finite Resolutions.

September 1987 - January 1988: Course for Operators at the Service of Trains, "Metro de Caracas C.A.", Caracas, Venezuela.

March 1986: First Workshop on Analysis, University of The Andes, College of Science, Department of Mathematics, Mérida, Venezuela.

August 1985: Summer School at the "Instituto Venezolano de Investigaciones Científícas" (Venezuelan Institute for Scientific Research), IVIC, Caracas, Venezuela.

Scholarships

April 2004 - August 2004: Scholarship from the Research Institute for Symbolic Computation (RISC-Linz), Johannes Kepler University, Linz, Austria.

March 2003 - August 2003: Scholarship from the Research Institute for Symbolic Computation (RISC-Linz), Johannes Kepler University, Linz, Austria.

Since Ocotber 2001: Scholarship from the University of The Andes, Mérida, Venezuela.

September 1974 - June 1987: Scholarship from the Ministery of Education of the Venezuelan Republic.

Languages

Spanich: Native.
English: Proficient.
German: Fair.