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Symbolic Differential Elimination for Symmetry Analysis

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ABSTRACT

Differential problems are ubiquitous in mathematical modeling of physical and scientific problems. Algebraic analysis of differential systems can help in determining qualitative and quantitative properties of solutions of such systems. In this tutorial paper we describe several algebraic methods for investigating differential systems.

Keywords: Lie symmetries, differential elimination, Gröbner bases.

1. INTRODUCTION

The idea of an algebraic approach to differential equations (DEs) has a long history. In the 19th century, Lie initiated the investigation of transformations, which leave a given differential equation invariant. Such transformations are commonly known as Lie symmetries. They form a group, a so-called Lie group. The basic idea here is to find a group of symmetries of a differential equations and then use this group to reduce the order or the number of variables appearing in the equation. Lie discovered that the knowledge of a one-parameter group of symmetries of an ordinary differential equation of order n allows us to reduce the problem of solving this equation to that of solving a new differential equation of order $n-1$ and integrating.

From the Riquier-Janet theory of PDEs at the beginning of the 20th century an algorithm emerged, the Janet bases algorithm, which is strikingly similar to the method of Gröbner bases for generating canonical systems for algebraic ideals as developed by Buchberger. By computing the Janet basis for the coefficients of the Lie

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symmetries of a differential equation, the determining system of these coefficients can be triangularized and ultimately solved. In fact, for linear systems of DEs we can directly apply Gröbner bases.

In symbolic treatment of DEs the ultimate goal should be a symbolic solution. However, this is rarely achieved. But it is also of great importance to decide whether a system of DEs is solvable. If there are solutions, then we can derive differential systems in triangular form such that the solutions of the original system are the (non-singular) solutions of the output system. Deriving symmetries helps in verifying numerical schemes for solution approximation. In case the given system consists of differential algebraic equations (DAEs) we may get a complete overview of the algebraic relations which the solutions must satisfy.

The importance of computer algebra tools in this field is enormous. It can be demonstrated by comparing the impact made by symmetry analysis and differential Galois theory. The latter one is a little known theory studied by a few pure mathematicians. The former remained in the same state for many decades following Lie's original work. The main reason for this historical factum is definitely the tedious determination of the symmetry algebra.

As soon as computer algebra systems emerged, the first packages to set up at least the determining equations were written. An effective symbolic treatment of differential problems depends crucially on algorithms in differential elimination theory. While the algebraic theory of elimination is well developed, for differential ideals, there are still many open problems. For instance, the membership problem or the ideal inclusion problems for finitely generated differential ideals are still not solved (compare [1]).

The aim of this tutorial paper is the symbolic, i.e., non-approximative, treatment of some aspects of differential elimination theory: differential Gröbner bases, involutive bases, characteristic sets, symmetry analysis.

2. MODEL DIFFERENTIAL EQUATIONS

The formulation of natural laws and of technological problems in the form of rigorous mathematical models is often given in terms of differential equations. Such equations relate the behavior of certain unknown functions (called *dependent variables*) at a given point (time, position, etc., called *independent variables*) to their behavior at neighboring points. In general, these equations hence contain derivatives of the dependent variables up to some finite order.

If the dependent variables are functions of a single variable, the equations are termed *ordinary differential equations* (ODEs for short). Examples of ODEs are presented in Section 2.1. If the unknown functions depend on several independent variables, then one deals with *partial differential equations* (PDEs for short). Examples of PDEs are given in Section 2.2.

2.1. Some ODE Models

Tables 1–3 present ordinary differential equations arising as simple models of some natural as well as ''everyday life''-physics and ecological phenomena. The first column in each table contains short descriptions of the phenomena via keywords. The second column contains ODE models for the phenomena described in the first column. Finally, the third column contains lists of the parameters arising in the ODE model equations from the second column, each with a short description of its ''real world'' interpretation.

Table 1. ODEs arising as simple models of *natural phenomena*. Here g denotes the gravitational constant $g \approx 9.81\,\mathrm{m/s^2}$.

Description of phenomenon	Model equation	Parameter interpretation
Free fall of a body near earth	$h''(t) = -g$	h . . . Height above ground
Free fall of a meteorid	$r''(t) = -gR^2/r^2$	r . . . Distance between earth and meteorid center
Developing drops	$k(r^3(t)v(t))' = -gr^3$	r . . . Radius of drop v . . . Velocity of drop k . . . Empirical constant
Falling rain	$mv'(t) = -mg + f(v)$ $v(t_0) = v_0$	$f(v)$ . . . Air resistance v . . . Velocity of drop m . . . Mass of drop
Motion of planets	$\mathbf{x}''(t) = -GM\mathbf{x}/r^3$	$\mathbf{x}$ . . . Position vector of planet $r = \|\mathbf{x}\|$, M . . . Mass of the sun G . . . Constant of gravitation
Cooling	$\tau'(t) = k(T - \tau)$	τ . . . Temp. of immersed body T . . . Temp. of cooling medium k . . . Material depend. constant

Table 2. ODEs arising as models of *ecological phenomena*.

Description of phenomenon	Model equation	Parameter interpretation
Population growth	$P'(t) = \alpha P - \beta P^2$	P . . . Population size, $\alpha, \beta > 0$ . . . Model constants.
Predator and prey	$u'(t) = (a - bv(t))u(t)$, $v'(t) = (ku(t) - l)v(t)$.	u . . . Prey species, v . . . Predator species, $a, b, k, l > 0$ . . . Model constants.
Competing species	$u'(t) = f(u(t), v(t))u(t)$, $v'(t) = g(u(t), v(t))v(t)$.	u . . . Competing species, v . . . Competing species, $a, b, k, l > 0$ . . . Model constants.

Table 3. ODEs arising as models of "*everyday life*"-*physics phenomena.*

Description of phenomenon	Model equation	Parameter interpretation
Cooling, pasteurization	$\tau'(t) = k(T - \tau)$.	τ . . . Temp. of immersed body, T . . . Temp. of cooling medium, k . . . Material depend. constant.
Outflow from a funnel	$h^{3/2}h'(t) + \lambda = 0$.	h . . . Height inside funnel, λ . . . Constant depending on physical parameters.
Heating and air conditioning	$\tau'(t) = k(T(t) - \tau) + H(t) + A(t)$.	T . . . Outside temperature, τ . . . Inside temperature, H . . . Temp. change from heating, A . . . Temp. change from air cond.
Electrical instruments	$CV'(t) = -I, V - L\frac{dI}{dt} = RI$.	I . . . Current of discharge, V . . . Voltage, R . . . Resistance, C . . . Condenser's capacity, L . . . Inductance of coil.
Mechanical vibrations	$my''(t) + ly'(t) + ky(t) = f(t)$.	m . . . Mass of particle, y . . . Displacement of particle, f . . . Total external force, $k, l > 0$ . . . Constants.
Collapse of driving shafts	$\mu u^{(4)}(x) = f$.	u . . . Shaft displacement, f . . . Centrifugal force density, μ . . . Constant dep. on material.

2.2. Some PDE Models

A body is *isotropic* if the thermal conductivity at each point in the body is independent of the direction of heat flow through the point. The temperature $u = u(x, y, z, t)$ in an isotropic body can be found by solving the partial differential equation

$$\partial_x(ku_x) + \partial_y(ku_y) + \partial_z(ku_z) = c\rho u_t,$$

where k, c, ρ are functions of (x, y, z). They represent thermal conductivity, specific heat, and density of the body at (x, y, z), respectively. When k, c, ρ are constants, this equation is known as the simple three-dimensional heat equation $u_{xx} + u_{yy} + u_{zz} = \frac{c\rho}{k} u_t$. If the boundary of the body is relatively simple, the solution to this equation can be found using Fourier series. An approach to the two-dimensional heat equation using symmetries can be found in the next section. The Poisson equation

$$u_{xx}(x, y) + u_{yy}(x, y) = f(x, y)$$

arises in the study of various time-independent physical problems such as the steady state distribution of heat in a plane region, the potential energy of a point in a plane acted on by gravitational forces in the plane, and two-dimensional steady-state problems involving incompressible fluids. If $f = 0$ and the temperature within the region R is determined by the temperature distribution on the boundary ∂R, the constraints are called Dirichlet boundary conditions, given by $u(x, y) = g(x, y)$ for all (x, y) on ∂R, see Figure 1.

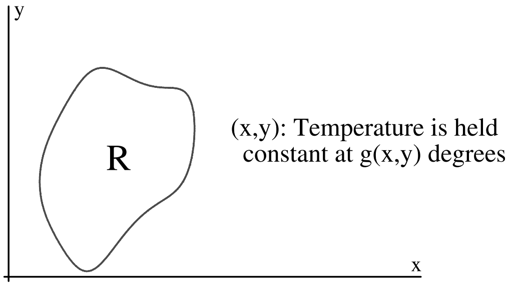


Fig. 1. Region R with $u(x, y) = g(x, y)$ on ∂R.

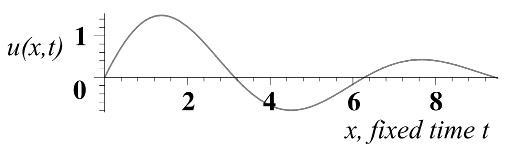


Fig. 2. Elastic string stretched between two supports.

Table 4. Linear PDEs arising as models of physical phenomena.

Description of phenomenon	Model equation	Remarks
Small transversal vibrations of strings	$u_{tt} = ku_{xx}$	D'Alembert equation
Propagation of light disturbances	$u_{tt} = k^2(u_{xx} + u_{yy} + u_{zz})$	Three-dimensional wave equation
Small transversal vibrations of uniform slender rods	$u_{tt} + \mu u_{xxxx} = f$	$\mu > 0, f \ldots$ Total force acting on rod
Thermal diffusion	$u_t = k^2 u_{xx}$	One-dimensional heat conduction equation
Stock option pricing	$u_t + \frac{1}{2}Ax^2 u_{xx} + Bxu_x = Cu$	Black-Scholes equation $A, B, C \ldots$ parameters

Table 5. Several *classical PDEs* (including non-linear ones) along with their names.

Equation name	Equation
Nonlinear heat conduction equation	$u_t = K(u)u_{xx} + K'(u)u_x^2, K \in C^1(\mathbb{R})$
Wave equation for an inhomogeneous medium	$u_{tt} = c(x)^2 u_{xx}, c \in C^1(\mathbb{R})$
Biharmonic equation	$-u_{tttt} = 2u_{xxtt} + u_{xxxx}$
Burgers equation	$u_x = uu_t + u_{tt}$
Korteweg-de Vries equation	$u_x = uu_t + u_{ttt}$
sine-Gordon equation	$u_{xt} = \sin(u)$

The one-dimensional wave equation

$$\alpha^2 u_{xx}(x,t) = u_{tt}(x,t)$$

models the vertical displacement $u(x,t)$ at point x at time t of an elastic string of length l stretched between two supports at the same horizontal level, provided that damping effects are neglected and the amplitude is not too large, see Figure 2.

The dependant variable $u = u(x,y,z,t)$ stands for physical quantities like *wave displacement* or *temperature*, whereas the independent variables x,y,z,t denote the *space* and *time* coordinates, respectively. Table 4 shows several linear PDEs arising as simple models of physical phenomena, and Table 5 presents some classical PDEs (including non-linear ones) together with their names.

3. GROUP ANALYSIS OF DIFFERENTIAL EQUATIONS

In this section, we present the method of group analysis of DEs by demonstrating its use in simplifying and integrating ODEs and PDEs.

We first introduce basic notions for symmetries of ODEs. These concepts extend to the case of partial differential equations, too.

3.1. Symmetries of ODEs

We introduce *transformation groups* and their *differential invariants*, which determine the *invariant equations* corresponding to the group. The differential invariants are solutions of a system of PDEs, called *system of differential invariants*.

Transformation Groups of Differential Equations Introducing new variables into a given DE is a widely used method in order to facilitate the solution process. Usually this is done in an ad hoc manner without guaranteed success. In particular, there is no criterion to decide whether a certain class of transformations will lead to an integrable equation or not. A critical examination of these methods was the starting point for Lie's symmetry analysis. We will now have a look on the behavior of DEs under special kind of transformations.

Let an ODE of order n be given as

$$\omega(x, y, y', \ldots, y^{(n)}) = 0. \tag{1}$$

The general solution of such an equation is a set of curves in the x-y-plane depending on n parameters $C_1, \ldots, C_n$, given by

$$\Theta(x, y, C_1, \ldots, C_n) = 0. \tag{2}$$

Invertible analytic transformations between two planes with coordinates (x, y) and (u, v), respectively, that are of the form

$$u = \sigma(x, y), \qquad v = \rho(x, y), \tag{3}$$

are called *point transformations*. We will encounter them in the form of *one-parameter groups of point transformations*

$$u = \sigma(x, y, \varepsilon), \qquad v = \rho(x, y, \varepsilon). \tag{4}$$

Here the real parameter ε ranges over an open interval I including 0, such that for any fixed value of ε, Equation (4) represents a point transformation. In addition, there exists a real group composition Φ such that

$$\begin{aligned} &\bar{x} = \sigma(x, y, \varepsilon), \qquad \bar{y} = \rho(x, y, \varepsilon), \qquad \bar{\bar{x}} = \sigma(\bar{x}, \bar{y}, \bar{\varepsilon}), \qquad \bar{\bar{y}} = \rho(\bar{x}, \bar{y}, \bar{\varepsilon}) \\ &\quad \Longrightarrow \quad \bar{\bar{x}} = \sigma(x, y, \Phi(\varepsilon, \bar{\varepsilon})), \qquad \bar{\bar{y}} = \rho(x, y, \Phi(\varepsilon, \bar{\varepsilon})), \end{aligned}$$

where $\bar{\varepsilon} \in I$ is such that $\Phi(\varepsilon, \bar{\varepsilon}) \in I$. Group transformations of this kind may be reparametrized such that we have $\Phi(\varepsilon, \bar{\varepsilon}) = \varepsilon + \bar{\varepsilon}$, and such that $\varepsilon = 0$ represents the identity element.

An Equation (1) is said to be *invariant* under the change of variables (3) where $v \equiv v(u)$, if it retains its form under this transformation, i.e., if the functional dependence of the transformed equation on u and v is the same as in the original Equation (1). Such a transformation is called a *symmetry* of the DE. The same transformation acts on the curves (2). If it is a symmetry, the functional dependence of the transformed curves of u and v must be the same as in Equation (2). This is not necessarily true for the parameters $C_1, \ldots, C_n$ because they do not occur in the DE itself. This means that the entirety of curves described by the two equations is the same, to any fixed values for the constants however may correspond a different curve in either set. In other words the solution curves are permuted among themselves by a symmetry transformation. It is fairly obvious that all symmetry transformations of a given DE form a group, the *symmetry group* of that equation.

Infinitesimal Generators and Prolongations Let a curve in the (x-y)-plane described by $y = f(x)$ be transformed under a point transformation of the form (3) into $v = g(u)$. Now the question arises of how the derivative $y' = df/dx$ corresponds to $v' = dg/du$ under this transformation. A simple calculation leads to the *first prolongation*

$$v' = \frac{dv}{du} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'} \equiv \chi(x, y, y').$$

Note that the knowledge of (x, y, y') and the equations of the point transformation (3) already determine v' uniquely, the knowledge of the equation of the curve is not required. This may be expressed by saying that the line element (x, y, y') is transformed into the line element (u, v, v') under the action of a point transformation. Similarly, the transformation law for derivatives of second order is obtained as

$$v'' = \frac{dv'}{du} = \frac{\chi_x + \chi_y y' + \chi_{y'} y''}{\sigma_x + \sigma_y y'}.$$

For later applications it would be useful to express the second derivative v'' explicitly in terms of σ and ρ. We do not give this more lengthy formula here, but instead provide the prolongation formulas for one-parameter groups of point transformations of the form

$$u = \sigma(x, y, \varepsilon), \qquad v = \rho(x, y, \varepsilon). \tag{5}$$

Here the transformation properties of the derivatives may be expressed in terms of the prolongation of the corresponding *infinitesimal generator*

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y, \tag{6}$$

where

$$\xi(x, y) = \frac{d}{d\varepsilon}\sigma(x, y, \varepsilon)\bigg|_{\varepsilon=0}, \qquad \eta(x, y) = \frac{d}{d\varepsilon}\rho(x, y, \varepsilon)\bigg|_{\varepsilon=0}.$$

The n-th prolongation of (6) is now defined in terms of the operator of total differentiation w.r.t. x, denoted by $D = \partial_x + \sum_{k=1}^{\infty} y^{(k)}\partial_{y^{(k-1)}}$ as

$$\begin{aligned} X^{(n)} &= X + \sum_{k=1}^{n} \zeta^{(k)}\partial_{y^{(k)}}, \text{ where} \\ \zeta^{(1)} &= D(\eta) - y' D(\xi), \\ \zeta^{(k)} &= D(\zeta^{(k-1)}) - y^{(k)} D(\xi) \quad \text{for } k = 2, 3, \ldots \end{aligned}$$

We give the two lowest ζ's explicitly:

$$\zeta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2,$$
$$\zeta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y''.$$

These two innocent looking expressions should not distract from the fact that the number of terms in $\zeta^{(k)}$ grows roughly as 2^k. But $\zeta^{(k)}$ is at least linear and homogeneous in $\xi(x, y)$ and $\eta(x, y)$ and its derivatives up to order k. For $k > 1$, $y^{(k)}$ occurs linearly and y' occurs with power $k + 1$ in $\zeta^{(k)}$.

Differential Invariants of Point Transformations Any r-parameter Lie transformation group may be represented by r infinitesimal generators

$$X_i = \xi_i \partial_x + \eta_i \partial_y, \quad i = 1, \ldots, r. \tag{7}$$

Any ordinary DE of order m with this r-parameter Lie group as symmetry group has to vanish under all m-th prolongations of the generators (7) and vice versa, i.e., this DE $\Phi \equiv \Phi(x, y, y', y'', \ldots)$ is a solution of the following system of linear homogeneous first order partial differential equations:

$$X_i^{(m)}\Phi = 0, \quad i = 1, \ldots, r, \tag{8}$$

The system (8) is called *system of differential invariants*; its fundamental solutions are called the *differential invariants* of the respective Lie group. Lie has discussed these systems in detail: for a recent presentation see [2].

The group property guarantees that Equation (8) is a complete system for Φ with $m + 2 - r$ solutions. It may be brought into Jacobian normal form, an analogue of the triangular form for matrices, before attempting to solve it. The dependencies of the fundamental solutions may then be chosen such that

$$\begin{aligned} \Phi_1 &\equiv \Phi_1(x, y, y', \ldots, y^{(r-1)}), \\ \Phi_2 &\equiv \Phi_2(x, y, y', \ldots, y^{(r)}), \\ &\vdots \\ \Phi_{m-r+2} &\equiv \Phi_{m-r+2}(x, y, y', \ldots, y^{(m)}). \end{aligned}$$

The invariants are linear in the highest derivative.

Example We consider the following transformation group that acts on the (x, y)-plane which is represented by

$$g = \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y\}.$$

Prolongation of its three generators up to the third order yields the following system of differential invariants (8):

$$\Phi_x = 0,$$
$$2x\Phi_x + y\Phi_y - y'\Phi_{y'} - 3y''\Phi_{y''} - 5y'''\Phi_{y'''} = 0,$$
$$x^2\Phi_x + xy\Phi_y - (y'x - y)\Phi_{y'} - 3y''x\Phi_{y''} - (5y'''x + 3y'')\Phi_{y''} = 0.$$

Using some strategy for solving systems of linear PDEs, for example, iterated narrowing transformations or *elimination*, we may arrive at the following two fundamental solutions:

$$\Phi_1 \equiv y''y^3, \qquad \Phi_2 \equiv y'''y^5 + 3y''y'y^4.$$

The DEs of order not higher than three that have the respective Lie group g as symmetry group have the general form $\omega(\Phi_1, \Phi_2)$ for some differentiable function ω.

3.2. Symmetries of PDEs

Finding differential invariants is accomplished in analogy to the ordinary case: the group generators have to be prolongated to the desired order; the prolongations are then interpreted as a system of linear PDEs whose fundamental solutions provide a basis of differential invariants.

We introduce the prolongation formulas that apply to the case of partial differential equations with one dependent variable u and n independent variables $x = x_1, \ldots, x_n$ (compare [3]). Partial derivatives $\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} u$ are represented by formal variables $u_{i_1 \ldots i_k}$, called *differential indeterminates*. They are symmetric in their indices. The differential variables of order k are denoted by $u^{(k)}$. We also use the convention to sum over the range of multiply occurring indices in products, e.g., $(D_i\xi_j)u_j = \sum_{j=1}^{n}(D_i\xi_j)u_j$.

The one-parameter Lie group of transformations in the parameter ε

$$x_i^* = X_i(x, u; \varepsilon) = x_i + \varepsilon\xi_i(x, u) + O(\varepsilon^2), \tag{9}$$

$$u^* = U(x, u; \varepsilon) = u + \varepsilon\eta(x, u) + O(\varepsilon^2), \tag{10}$$

$i = 1, 2, \ldots, n$, acting on (x, u)-space has as its infinitesimal generator

$$X = \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u.$$

The k-th extension of Equation (9), (10), given by

$$
\begin{aligned}
x_i^* &= X_i(x, u; \varepsilon) = x_i + \varepsilon \xi_i(x, u) + O(\varepsilon^2), \\
u^* &= U(x, u; \varepsilon) = u + \varepsilon \eta(x, u) + O(\varepsilon^2), \\
&\vdots \\
u^*_{i_1 i_2 \dots i_k} &= U_{i_1 i_2 \dots i_k}(x, u, u^{(1)}, \dots, u^{(k)}; \varepsilon) \\
&= u_{i_1 i_2 \dots i_k} + \varepsilon \eta^{(k)}_{i_1 i_2 \dots i_k}(x, u, u^{(1)}, \dots, u^{(k)}) + O(\varepsilon^2),
\end{aligned}
$$

where $i = 1, 2, \dots, n$ and $i_l = 1, 2, \dots, n$ for $l = 1, 2, \dots, k$ with $k = 1, 2, \dots$, has as its k-th extended infinitesimal generator

$$
X^{(k)} = \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u + \eta_i^{(1)}(x, u, u^{(1)})\partial_{u_i} + \cdots + \eta^{(k)}_{i_1 i_2 \dots i_k}\partial_{u_{i_1 i_2 \dots i_k}},
$$

$k = 1, 2, \dots$; explicit formulas for the extended infinitesimals $\{\eta^{(k)}\}$ are given recursively by

$$
\eta_i^{(1)} = D_i \eta - (D_i \xi_j) u_j, \quad i = 1, 2, \dots, n, \tag{11}
$$

$$
\eta^{(k)}_{i_1 i_2 \dots i_k} = D_{i_k} \eta^{(k-1)}_{i_1 i_2 \dots i_k} - (D_{i_k} \xi_j) u_{i_1 i_2 \dots i_{k-1} j}, \tag{12}
$$

$i_l = 1, 2, \dots, n$ for $l = 1, 2, \dots, k$ with $k \geq 2$.

4. EXAMPLES

The two following examples demonstrate the use of symmetries on ODEs; key ingredients are canonical coordinates and Lie's integration algorithm, see also [4].

4.1. A First Order ODE

We demonstrate how to reduce the order of a first-order ODE with the help of a symmetry. This results in integration. We use the method of canonical coordinates.

Example (Canonical Coordinates) We consider the Riccati equation

$$
y' + y^2 - \frac{2}{x^2} = 0. \tag{13}
$$

It is invariant under the group of transformations

$$
\overline{x} = x e^{\varepsilon}, \qquad \overline{y} = y e^{-\varepsilon} \qquad (, \overline{y}' = y' e^{-2\varepsilon}). \tag{14}
$$

Its infinitesimals $(\frac{d}{d\epsilon}\overline{x}, \frac{d}{d\epsilon}\overline{y})_{\epsilon=0} = (x, -y)$ determine the infinitesimal symmetry

$$X = x\partial_x - y\partial_y.$$

Canonical coordinates t, u for Equation (13) are obtained by solving $X(t) = 1$, $X(u) = 0$ and have the form

$$t = \ln|x|, \qquad u = xy.$$

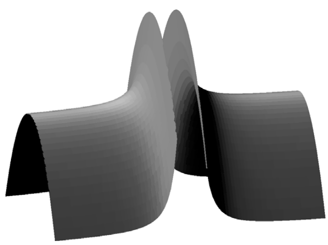
In these coordinates, the inhomogeneous stretchings (14) are replaced by the translation group

$$\overline{t} = t + \varepsilon, \qquad \overline{u} = u, \qquad \overline{u}' = u'$$

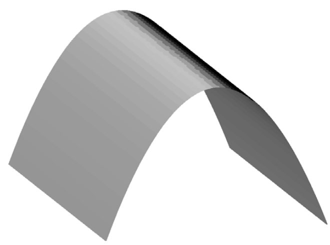
and Equation (13) takes the integrable form

$$u' + u^2 - u - 2 = 0. \tag{15}$$

Geometrically, the frame of Equation (15) is now a ‘‘straightened out’’ parabolic cylinder. In general, the frame of a first order ODE $y' = f(x, y)$ is the surface in the space of three independent variables, x, y, and p, given by $p = f(x, y)$.



frame of Riccati’s equation



and its transform

Analytically, we note that Equation (15) does not depend on t explicitly. Integrating Equation (15) gives

$$\ln\left|\frac{u+1}{u-2}\right| - 3t = \text{const.},$$

provided that $u + 1 \neq 0$ and $u - 2 \neq 0$. Substituting the expressions for t and u in terms of x and y, one arrives at the solution

$$y = \frac{2x^3 + C}{x(x^3 - C)}, \quad C = \text{const.},$$

provided that $xy - 2 \neq 0$ and $xy + 1 \neq 0$. In case these expressions are zero, one arrives at $y_1 = 2/x$ and $y_2 = -1/x$, respectively.

4.2. A Second Order ODE

If a second order ODE $y'' = f(x, y, y')$ admits one symmetry, its order may be reduced by one. In case it admits two symmetries, integration can be achieved. Reduction of order and successive integration are applicable to higher order equations as well. The restriction to second order is essential, however, for the method of integration using canonical forms of two-dimensional Lie algebras, see [4]. These canonical forms and their invariant second-order equations are presented in Table 6. For $X_i = \xi_i \partial_x + \eta_i \partial_y$, we denote the wedge-product of X_1, X_2 by $X_1 \vee X_2 := \xi_1 \eta_2 - \xi_2 \eta_1$.

Based on this classification, we sketch *Lie's integration algorithm* for integrating second-order ODEs that admit a two-dimensional Lie algebra in Table 7.

Example (Lie's integration algorithm) We consider the second order ODE

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}. \tag{16}$$

Step 1. The *calculation of its admissible Lie algebra* is demonstrated in Subsection 5, yielding two linearly independent operators.

$$X_1 = x^2 \partial_x + xy \partial_y, \qquad X_2 = x \partial_x + \frac{y}{2} \partial_y. \tag{17}$$

Table 6. Canonical forms of two-dimensional Lie algebras and their invariant second-order equations.

Type	L_2 structure		Basis of L_2		Invariant equation
	$[X_1, X_2]$	$X_1 \vee X_2$	X_1	X_2	
I	0	$\neq 0$	∂_x	∂_y	$y'' = f(y')$
II	0	0	∂_y	$x\partial_y$	$y'' = f(x)$
III	X_1	$\neq 0$	∂_y	$x\partial_x + y\partial_y$	$y'' = \frac{1}{x} f(y')$
IV	X_1	0	∂_y	$y\partial_y$	$y'' = f(x) y'$

Table 7. Lie's integration algorithm.

Step	Action	Result
1.	Compute admitted Lie Algebra L_r.	basis $X_1, \ldots, X_r$.
2.	If $r = 2$ go to step 3. If $r > 2$ distinguish any 2-dimensional subalgebra L_2 of L_r.	basis X_1, X_2 for L_2.
3.	Determine type of L_2 according to table; eventually choose a new basis X_1', X_2'.	canonical form.
4.	Go over to canonical variables x, y. Rewrite equation in these variables and integrate it.	change of variables.
5.	Rewrite solution in terms of original variables.	solution.

According to the algorithm, we advance directly to the third step.

Step 3. To *determine the type of the Lie algebra*, we consider

$$[X_1, X_2] = -X_1, \qquad X_1 \vee X_2 = -\frac{x^2 y}{2} \neq 0.$$

After merely changing the sign of X_2, the basis has exactly the structure of type *III* in the canonical form table.

Step 4. To *determine an integrating change of variables*, we first introduce canonical variables for X_1 as the solutions of $X_1(t) = 1$ and $X_1(u) = 0$. They are given by

$$t = \frac{y}{x}, \qquad u = -\frac{1}{x},$$

transforming the operators to

$$X_1 = \partial_u, \qquad X_2 = \frac{t}{2}\partial_t + u\partial_u.$$

This is basically still type *III*; the factor $\frac{1}{2}$ in X_2 does not hinder integration. Excluding the solution $y = Kx$, the equation written in the new variables is

$$u'' + \frac{1}{t^2}u'^2 = 0.$$

Integrating once, we get $u' = t/(C_1 t - 1)$. Hence

$$u = -\frac{t^2}{2} + C \quad \text{for } C_1 = 0, \text{ or}$$
$$u = \frac{t}{C_1} + \frac{1}{C_1^2}\ln|C_1 t - 1| + C_2 \quad \text{for } C_1 \neq 0.$$

Step 5. The solutions in the original variables are then

$$y = Kx, \qquad y = \pm\sqrt{2x + Cx^2},$$
$$0 = C_1 y + C_2 x + x \ln\left|C_1 \frac{y}{x} - 1\right| + C_1^2.$$

4.3. Two Second Order PDEs

In this subsection we present the calculation of symmetries and their use in finding invariant solutions of second order PDEs. This time, we solve the determing system (of the second example) ‘by hand’ – compare also [3].

The Heat Equation The heat equation

$$z_{xx} - z_y = 0 \tag{18}$$

is an example of a second order PDE by which we demonstrate the computation of symmetry generators and their use in finding invariant solutions. In analogy to Subsection 3.1, a necessary and sufficient condition for an infinitesimal generator

$$X = \xi_1(x, y, z)\partial_x + \xi_2(x, y, z)\partial_y + \eta(x, y, z)\partial_z \tag{19}$$

to be admitted by Equation (18) is

$$X^{(2)}(z_{xx} - z_y) = 0 \text{ mod } z_{xx} = z_y, \tag{20}$$

where we replace any occurrence of z_{xx} by z_y after application of the operator $X^{(2)}$. The operator $X^{(2)}$ is the second order prolongation of X and given by

$$X^{(2)} = \xi_1\partial_x + \xi_2\partial_y + \eta\partial_z + \eta_1^{(1)}\partial_{z_x} + \eta_2^{(1)}\partial_{z_y} + \eta_{11}^{(2)}\partial_{z_{xx}} + \eta_{12}^{(2)}\partial_{z_{xy}} + \eta_{22}^{(2)}\partial_{z_{yy}},$$

where $\eta_1^{(1)}, \eta_2^{(1)}, \eta_{11}^{(2)}, \eta_{12}^{(2)}, \eta_{22}^{(2)}$ are defined as in Subsection 3.1. The determining equation for Equation (18) is

$$\eta_{11}^{(2)} - \eta_2^{(1)} = 0 \text{ mod } z_{xx} = z_y. \tag{21}$$

We treat Equation (21) as $\eta_{11}^{(2)} - \eta_2^{(1)} = 0$, where every occurrence of z_{xx} is replaced by z_y. This equation is polynomial in $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$, and since ξ_1, ξ_2, η only depend on x, y, z, we may equate the coefficients of $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ (and their powers) in (20) to zero. The result is an overdetermined system of linear homogeneous equations in ξ_1, ξ_2, η and their partial derivatives up to order two, called *determining system*.

The procedure outlined above holds in general. We demonstrate how to solve such a system in the next example. The solution gives the Lie algebra spanned by the following six generators, each of which corresponds to a one-parameter group:

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = x\partial_x + 2y\partial_y,$$
$$X_4 = 4xy\partial_x + 4y^2\partial_y - (x^2 + 2y)z\partial_z, \qquad X_5 = 2y\partial_x - xz\partial_z, \qquad X_6 = z\partial_z.$$

Let us consider the infinitesimal generator X_4, which corresponds to the parameter c_1. The one-parameter Lie group of transformations

$$\bar{x}(x, y, z, \epsilon), \qquad \bar{y}(x, y, z, \epsilon), \qquad \bar{z}(x, y, z, \epsilon) \tag{22}$$

corresponding to $X_4 = 4xy\partial_x + 4y^2\partial_y - (x^2 + 2y)z\partial_z$ is obtained by solving the initial value problem

$$(\bar{x}, \bar{y}, \bar{z})[\epsilon = 0] = (x, y, z) \tag{23}$$

for the following first order system of ODEs:

$$\frac{d\bar{x}}{d\varepsilon} = 4\bar{x}\bar{y}, \tag{24}$$

$$\frac{d\bar{y}}{d\varepsilon} = 4\bar{y}^2, \tag{25}$$

$$\frac{d\bar{z}}{d\varepsilon} = -(\bar{x}^2 + 2\bar{y})\bar{z}. \tag{26}$$

The solution of Equation (25) is $\bar{y} = \frac{1}{C-4\epsilon}$, and by Equation (23) we obtain

$$\bar{y}(x, y, z, \epsilon) = \frac{y}{1 - 4\epsilon y}. \tag{27}$$

By this and Equation (24) we get $\bar{x} = \frac{C}{1-4\epsilon y}$, and by Equation (23) we obtain

$$\bar{x}(x, y, z, \epsilon) = \frac{x}{1 - 4\epsilon y}. \tag{28}$$

Similarly, by Equations (28), (27), (26) and (23) we obtain

$$\bar{z}(x, y, z, \epsilon) = z\sqrt{1 - 4\epsilon y}\exp\left(-\frac{\epsilon x^2}{1 - 4\epsilon y}\right). \tag{29}$$

Every invariant solution $z = \Phi(x, y)$ of Equation (18) corresponding to X_4 satisfies

$$X_4(z - \Phi(x, y)) = 0 \text{ when } z - \Phi(x, y),$$

i.e.,

$$4xy\frac{\partial\Phi}{\partial x} + 4y^2\frac{\partial\Phi}{\partial y} = -(x^2 + 2y)\Theta. \tag{30}$$

We solve Equation (30) by solving the corresponding characteristic equation

$$\frac{dx}{4xy} = \frac{dy}{4y^2} = \frac{dz}{-(x^2 + 2y)z}$$

which has the two invariants

$$\frac{x}{y} \quad \text{and} \quad z\sqrt{y}e^{x^2/4y}.$$

The solution of Equation (18) is now defined by the invariant form

$$z\sqrt{y}e^{x^2/4y} = \phi\left(\frac{x}{y}\right),$$

or, in explicit form,

$$z = \Phi(x, y) = \frac{1}{\sqrt{y}} e^{-x^2/4y} \phi(\zeta), \tag{31}$$

where $\zeta = \frac{x}{y}$ is the similarity variable and ϕ denotes an arbitrary twice differentiable function. Substitution of Equation (31) into Equation (18) leads to $\phi''(\zeta) = 0$. Hence, invariant solutions of Equation (18) resulting from X_4 are of the form

$$z = \Phi(x, y) = \frac{1}{\sqrt{y}} e^{-x^2/4y} \left\{ C_1 + C_2 \frac{x}{y} \right\}.$$

For any solution $z = \Phi(x, y)$ of Equation (18), that is not invariant under X_4, we find a one-parameter family of solutions $z = \phi(x, y, \epsilon)$ generated by X_4: Let

$$\begin{aligned} x^* &= \bar{x}(x, y, z, \epsilon) = \frac{x}{1 - 4\epsilon y}, \\ y^* &= \bar{y}(x, y, z, \epsilon) = \frac{y}{1 - 4\epsilon y}, \\ z^* &= \Phi(\bar{x}, \bar{y}). \end{aligned}$$

By $\bar{z}(\cdot, \cdot, \cdot, -\epsilon)$ we denote the third component of the inverse transformation corresponding to X_4. Then $z = \phi(x, y, \epsilon) = \bar{z}(x^*, y^*, z^*, -\epsilon) =$

$$\Phi\left(\frac{x}{1 - 4\epsilon y}, \frac{y}{1 - 4\epsilon y} \right) \frac{1}{\sqrt{1 - 4\epsilon y}} \exp\left(\frac{\epsilon x^2}{1 - 4\epsilon y} \right).$$

Wave Equation for an Inhomogeneous Medium We consider the wave equation for a variable wave speed $c(x)$:

$$z_{yy} = c(x)^2 z_{xx}. \tag{32}$$

It is a linear PDE and hence (see [5, Sec. 6]) can only admit infinitesimal generators of the form

$$X = \xi_1(x, y)\partial_x + \xi_2(x, y)\partial_y + f(x, y) z \partial_z.$$

In analogy to the previous example we obtain the invariance condition

$$\eta_{22}^{(2)} = c(x)^2 \eta_{11}^{(2)} + 2c(x)c'(x)\xi_1 z_{xx} \text{ when } (32).$$

The resulting determining system is

$$(\xi_1)_y - c(x)^2 (\xi_2)_x = 0, \tag{33}$$

$$c(x)[(\xi_2)_y - (\xi_1)_x] + c'(x)\xi_1 = 0, \tag{34}$$

$$(\xi_2)_{yy} - c(x)^2 (\xi_2)_{xx} - 2f_y = 0, \tag{35}$$

$$(\xi_1)_{yy} + c(x)^2 [2f_x - (\xi_1)_{xx}] = 0, \tag{36}$$

$$f_{yy} - c(x)^2 f_{xx} = 0. \tag{37}$$

Solving Equation (33) for $(\xi_2)_x$ and (34) for $(\xi_2)_y$ and setting $(\xi_2)_{xy} = (\xi_2)_{yx}$ we find

$$(\xi_1)_{xx} - (\xi_1)_{yy}/c(x)^2 - (\xi_1 H(x))_x = 0, \tag{38}$$

where $H(x) = c'(x)/c(x)$. Solving Equations (38) and (36) leads to

$$f(x, y) = \frac{1}{2} H(x)\xi_1(x, y) + S(y), \tag{39}$$

where $S(y)$ is an arbitrary function of y. Substituting Equation (39) into Equation (35) and then solving (33) for $(\xi_1)_y$ and (34) for $(\xi_1)_x$ and setting $(\xi_1)_{xy} = (\xi_1)_{yx}$, we find that $S(y) = const = s$, so that $f = \frac{1}{2}H\xi_1 + s$. Substituting f in Equation (37) and using Equation (36) we get $H''\xi_1 + 2H'(\xi_1)_x + H(H\xi_1)_x = 0$ or, equivalently,

$$[(2H' + H^2)(\xi_1)^2]_x = 0.$$

We now only consider the case $2H' + H^2 = 0$. Then

$$c(x) = (Ax + B)^2,$$

where A, B are arbitrary constants. Then $H(x) = \frac{2A}{Ax+B}$. For any solution $\xi_1(x, y)$ of Equation (38), one finds that $\xi_2(x, y), f(x, y)$ solving Equations (33)–(37) are given by:

$$\xi_2(x, y) = \int [(\xi_1)_x - H\xi_1] dy,$$
$$f(x, y) = \frac{A\xi_1(x, y)}{Ax + B}.$$

So $\{\xi_1, \xi_2, f\}$ determine a non-trivial infinite-parameter Lie group for

$$z_{yy} = (Ax + B)^4 z_{xx}. \tag{40}$$

If $A \neq 0$ this equation can be transformed to the wave equation

$$\bar{z}_{\bar{x}\bar{y}} = 0$$

by the point transformation

$$\bar{x} = (Ax + B)^{-1} + Ay,$$
$$\bar{y} = (Ax + B)^{-1} - Ay,$$
$$\bar{z} = (Ax + B)^{-1} z.$$

The general solution of PDE (40) is then

$$z = (Ax + B)[F(\bar{x}) + G(\bar{y})],$$

where F, G are twice differentiable functions.

4.4. Literature and Implementations

The most complete work on *group analysis of ordinary differential* equations is still [7]. A very broad introduction and comprehensive reference for group analysis of differential equations in general is [8]. In handbook style, this series presents *newly developed theoretical and computational methods*, meeting the needs of the applied reader as well as those of the researcher. In Chapter 13 and 14 in Volume 3, the reader finds an account on *symbolic software for calculating symmetries* by Hereman. Table 8 is taken from [9].

The last four columns in this table indicate the scope of the programs: point symmetries, generalized symmetries, non-classical symmetries and whether the determining system can be solved automatically. Recent Maple programs for generating classical symmetries are DESOLV by Carminati and Vu [10], RIF by Reid and Wittkopf and SYMMETRIE by Hickman.

Finally, some text books for the more applied reader are [6, 11]. Hillgarter contributed to the symmetry classification problem for a special class of PDEs [12]. This work was inspired by Fritz Schwarz, whose expertise in the algorithmic aspects of the field is reflected in [2].

Table 8. Scope of Lie symmetry programs.

Name	System	Developer(s)	Point	Gen.	Non-class.	Solves Det. Eqs.
CRACK	REDUCE	Wolf & Brand	–	–	–	Yes
DELiA	Pascal	Bocharov et al.	Yes	Yes	No	Yes
DIFFGROB2	Maple	Mansfield	–	–	–	Reduction
DIMSYM	REDUCE	Sherring	Yes	Yes	No	Yes
LIE	REDUCE	Eliseev et al.	Yes	Yes	No	No
LIE	muMATH	Head	Yes	Yes	Yes	Yes
Lie	Mathematica	Baumann	Yes	No	Yes	Yes
LieBaecklund	Mathematica	Baumann	No	Yes	No	Interactive
LIEDF/INFSYM	REDUCE	Gragert &	Yes	Yes	No	Interactive
LIEPDE	REDUCE	Wolf & Brand	Yes	Yes	No	Yes
Liesymm	Maple	Carminati et al.	Yes	No	No	Interactive
MathSym	Mathematica	Herod	Yes	No	Yes	Reduction
NUSY	REDUCE	Nucci	Yes	Yes	Yes	Interactive
PDELIE	MACSYMA	Vafeades	Yes	Yes	No	Yes
SPDE	REDUCE	Schwarz	Yes	No	No	Yes
SYMCAL	Maple/MACSYMA	Reid & Wittkopf	–	–	–	Reduction
SYM_DE	MACSYMA	Steinberg	Yes	No	No	Partially
symgroup.c	Mathematica	Bérubé & de Montigny	Yes	No	No	No
SYMMGRP.MAX	MACSYMA	Champagne et al.	Yes	No	Yes	Interactive
SYMSIZE	REDUCE	Schwarz	–	–	–	Reduction

5. DIFFERENTIAL ELIMINATION

Several methods in polynomial elimination theory can be reformulated to also apply to ideals of differential polynomials, or they have first been defined for differential polynomials but have found successful application to algebraic polynomials.

Differential Gröbner bases appeared first in [13] with further developments in [14] and [15]. Unfortunately, differential Gröbner bases are generally infinite, so they do not provide a general solution of the differential ideal membership problem. It is even known that the general membership problem is undecidable [16]. If, however, a finite differential Gröbner basis is known, ideal membership can be tested effectively. Carrá-Ferro could show that differential ideals that are generated by finitely many linear differential polynomials have a finite differential Gröbner basis with respect to an orderly ranking.

For linear PDEs with polynomial coefficients it is also possible to use an extension of the ordinary polynomial Gröbner bases theory (see [17, 18] for an introduction to the polynomial case) to Weyl algebras in order to simplify overdetermined systems. Here the system is saturated by all integrability conditions. The Maple computer algebra system comes with the package `Groebner` which is able to compute Gröbner bases in Weyl algebras.

Take, for example, the Equation (16). In order to determine the Lie symmetry algebra, one starts with undetermined functions $\xi(x,y)$ and $\eta(x,y)$ for the infinitesimal generator

$$X := \xi\partial_x + \eta\partial_y$$

and first sets up the determining system, as described, for example, in [19]. Basically ξ and η have to satisfy an equation [identically for all x and y satisfying (16)] that is obtained by applying the second prolongation $X^{(2)}$ of X to the original Equation (16). Equating coefficients of higher order derivatives leads to the following equations for ξ and η.

$$\frac{\partial^2\xi}{\partial y^2} = 0 \tag{41}$$

$$y^2\frac{\partial^2\eta}{\partial y^2} - 2y^2\frac{\partial^2\xi}{\partial x\partial y} - 2\frac{\partial\xi}{\partial y} = 0 \tag{42}$$

$$2xy^3\frac{\partial^2\eta}{\partial x\partial y} - xy^3\frac{\partial^2\xi}{\partial x^2} - xy\frac{\partial\xi}{\partial x} + 3y^2\frac{\partial\xi}{\partial y} + 2x\eta = 0 \tag{43}$$

$$x^2y^2\frac{\partial^2\eta}{\partial x^2} + 2xy\frac{\partial\xi}{\partial x} - x^2\frac{\partial\eta}{\partial x} - xy\frac{\partial\eta}{\partial y} - y\xi - x\eta = 0 \tag{44}$$

This is a system of *linear* PDEs. A computation of a Gröbner basis (with respect to an appropriate elimination ranking) in the algebra of linear differential operators leads to the triangular system

$$\frac{\partial \xi}{\partial y} = 0 \tag{45}$$

$$x^2 \frac{\partial^2 \xi}{\partial x^2} - 2x \frac{\partial \xi}{\partial x} + 2\xi = 0 \tag{46}$$

$$2\eta - y \frac{\partial \xi}{\partial x} = 0 \tag{47}$$

which is much easier to solve than the original system of determining equations. As a general solution we get

$$\xi = C_1 x^2 + 2C_2 x, \qquad \eta = (C_1 x + C_2) y$$

from which the independent operators in Equation (17) are derived.

The above Gröbner basis computation is not directly performed on the differential expressions. Before starting the computation, the system is transformed to a system of polynomials in a module over a Weyl algebra by formally replacing $\frac{\partial}{\partial x}$ by a symbol X and $\frac{\partial}{\partial y}$ by a symbol Y. Furthermore ξ and η are replaced by the first and second unitvector e_1 and e_2, respectively. For example, Equation (42) is translated to

$$y^2 Y e_2 - 2y^2 XY e_1 - 2Y e_1.$$

Doing the translation for the whole differential system leads, however, not to a system in a polynomial module but rather to the module $W_2 \times W_2$ over the Weyl algebra W_2. A Weyl algebra of dimension n over a field K is the free associative K-algebra

$$W_n = K\langle x_1, \ldots, x_n, X_1, \ldots, X_n \rangle.$$

modulo the commutation rules

$$x_i x_j = x_j x_i, \qquad X_i X_j = X_j X_i, \qquad X_i x_k = x_k X_i, \qquad X_i x_i = x_i X_i + 1$$

for all $i, j, k \in \{1, \ldots, n\}$ with $i \neq k$.

In our case the skew-commutation rules $Xx = xX + 1$ and $Yy = yY + 1$ originate from the Leibniz rule. The non-commutative domain $W_2 \times W_2$, however, still has certain elimination properties and admits the computation of Gröbner bases [19].

By computing the determining equations by the package `DESOLV_R5` by Vu and Carminati (cf. [10]) the above computation can be done in Maple V 5.1.

```
read("Desolv_r5"):
deq:=D[1,1](y)(x) - D[1](y)(x)/y(x)^2 + 1/(x*y(x));
deteqs:=gendef([deq],[y],[x]):
dq := subs(diff=F,xi[x](x,y)=e1,eta[y](x,y)=e2,deteqs[1]);
F:=proc(a,b) cat('D',b)*a end:
with(Ore_algebra): with(Groebner):
W:=diff_algebra([Dx,x],[Dy,y], comm={e1,e2},polynom={e1,e2}):
T:=termorder(W,lexdeg([e1,e2],[Dx,Dy]),[e1,e2]):
gbasis(p,T);
```

The final output is

```
[ 2*e2-y*Dx*e1, Dy*e1, e1*Dx^2*x^2-2*Dx*e1*x+2*e1]
```

and is easily translated back to the triangular system (45)–(47).

Basically, a Gröbner basis computation is a systematic way of adding *all* integrability conditions and reducing them with existing relations. For more details we refer to the standard textbooks [17, 18]. The non-commutative case of algebras of solvable type is treated, for example, in [19, 20].

Usually, the system of determining equations contains a huge number of equations. Take, for example, the Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} + u\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^4 u}{\partial x^4} = 0.$$

For this fourth order equation we set up the equations in order to determine the coefficients ξ_1, ξ_2, and η of the general symmetry generator

$$X := \xi_1(x,t,u)\partial_x + \xi_2(x,t,u)\partial_t + \eta(x,t,u)\partial_u.$$

In analogy to the previous example we have to compute the fourth prolongation of X. It leads to a system of 47 equations which can be generated automatically, for example, by the Maple package DESOLV_R5 in the following way.

```
read("Desolv_r5"):
bq:=D[1,1,1,1](u)(x,t)+D[1](u)(x,t)^2 +
  u(x,t)*D[1,1](u)(x,t)+D[2,2](u)(x,t);
deteqs:=gendef([bq],[u],[x,t]):
```

The package immediately applies some simplifications to reduce the number of equations to 12 of order 4. The question arises of whether or not this system is consistent, i.e., whether there are solutions at all. In the linear case, Gröbner bases are one tool to decide this problem. The computation of a Gröbner basis of the determining equations of the Boussinesq equation with respect to an appropriate ranking leads to an easily solvable system of 10 equations of order 2. We find that the symmetry algebra is spanned by the three elements

$$v_1 = \partial_x \qquad v_2 = \partial_t \qquad v_3 = x\partial_x + 2t\partial_t - 2u\partial_u.$$

Gröbner bases are not the only tool for decisions and computations in differential elimination theory. The theory of *involutive bases* has its foundation in the theory of PDEs given by Riquier [21] and Janet [22, 23] at the beginning of the 20th century. From the observation that a closed form solution of any system of partial differential equations may only be obtained for exceptional cases they focused their study to restricted questions of whether a solution exists at all or how one could find its degree of arbitrariness. Their constructive approach to algebraic analysis of PDEs was later followed by Thomas [24] and more recently by Pommaret [25]. The main idea of the approach is rewriting the initial differential system into another, so-called involutive form so that all its integrability conditions are satisfied. In contrast to differential Gröbner bases, involutive bases are finite. Since an involutive basis has all integrability conditions included it is possible to compute a Taylor series expansion of an analytic solution in a straightforward way. From an involutive basis one can immediately read off the degree of arbitrariness of the solution, cf. [26].

Characteristic sets are due to Ritt [27] and have further been adapted to algebraic polynomials by Wu [28]. The main idea is to transform the equations into triangular form in such a way that the solutions stay the same. However, the ideal is not preserved in general; multiplicities of solutions can change.

6. CONCLUSION

As we have seen above, current computer algebra techniques provide a computational algebraic approach to the analysis of systems of differential equations and sometimes also to their solution. But despite all the success of symbolic methods in differential equations (Lie symmetries, differential Galois theory, Janet bases, differential Gröbner bases, etc.), these theories are not and probably never will be able to solve the majority of differential problems in engineering. However, with further research into this area we might be able to tackle simplified problems. Toy models that can be solved analytically are important for obtaining a deeper understanding of the underlying structures. A deeper understanding of such simplified problems may well lead to more efficient numerical algorithms for large problems.

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