ON SYMBOLIC PARAMETRIZATION OF ALGEBRAIC CURVES *

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Abstract. In this paper we give an overview of the symbolic computation of rational parametrizations for algebraic curves. Parametrization of algebraic curves has been a topic in algebraic geometry for a long time and there are basically two approaches to the problem. The first approach is a geometric one, whereby a parametrization is determined from the intersection points of the given curve with a certain linear system of algebraic curves. The second one is an algebraic one involving the computation of integral bases. Our own research has mainly been on the first of these approaches, so will stress the geometric approach.

1. Introduction.

Let $\mathbf{K}$ be a computable field of characteristic zero, and let $F(x_1, x_2, x_3) \in \mathbf{K}[x_1, x_2, x_3]$ be a homogeneous polynomial defining an algebraic curve $\mathcal{C}$ in $\mathbf{P}^2(\mathcal{K})$, the projective plane over the algebraic closure $\mathcal{K}$ of $\mathbf{K}$. We call $\mathbf{K}$ the *field of definition* of $\mathcal{C}$. The curve $\mathcal{C}$ is *rational or parametrizable over the field* $\mathbf{L}$, $\mathbf{K} \subseteq \mathbf{L} \subset \mathcal{K}$, if and only if there exist polynomials $\varphi(t), \psi(t), \chi(t) \in \mathbf{L}[t]$ such that

(1) for almost every $t_0 \in \mathcal{K}$ (i.e. for every but a finite number of exceptions), $(\varphi(t_0), \psi(t_0), \chi(t_0))$ is a point on $\mathcal{C}$, and

(2) for almost every point (x_0, y_0, z_0) on $\mathcal{C}$ there is a $t_0 \in \mathcal{K}$ such that $(x_0, y_0, z_0) = (\varphi(t_0), \psi(t_0), \chi(t_0))$.

If φ, ψ, χ satisfy the conditions (1) and (2), we call (φ, ψ, χ) a (*rational*) *parametrization of* $\mathcal{C}$ *over* $\mathbf{L}$, and we call $\mathbf{L}$ the *field of parametrization.* A curve $\mathcal{C}$ is *rational* if and only if it has a rational parametrization. The rational curves are exactly the irreducible curves of genus zero. A parametrization is called *proper* if and only if almost every point on C is generated by exactly one value of the parameter.

The problem of parametrization has been treated implicitly already in [4]. An outline of a parametrization algorithm can be found in [11]. [7] gives a method for transforming any parametrization into a proper one. The approach described in [2] does not discuss the problems of algebraic numbers in parametrization, so the treatment is basically a numerical one. Symbolic algorithms for parametrization are described in [8], [6], and [10]. The complexity of the algorithm in [8] has been analyzed in [5], it turns out to be polynomial in the input. The parametrization algorithm described in [8] has been implemented in the CASA system for symbolic computation in algebraic geometry, an early version of which is described in [3].

There are basically two approaches to the symbolic parametrization of an algebraic curve $\mathcal{C}$ defined by a polynomial $f(x, y)$ over $\mathbf{Q}$ or some other field $\mathbf{K}$ of characteristic 0.

In the geometric approach, we construct a linear system of curves of dimension 1, such that all the elements of the system have their intersection points with $\mathcal{C}$ fixed except for one "free" intersection point. By varying the parameter in the 1–dimensional linear system of curves we let the free intersection point move along $\mathcal{C}$, thus creating a rational dependence of the points on $\mathcal{C}$ on the parameter of the system. In constructing the linear system of curves we might need to introduce algebraic numbers, i.e. enlarge the field of definition $\mathbf{Q}$ to an algebraic extension $\mathbf{Q}(\alpha)$. From [4] we can deduce that this algebraic extension can be kept small. In fact, for a curve of odd degree

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we need no algebraic extension of the ground field, whereas for a curve of even degree we might need an algebraic extension of degree 2. The crucial problem for symbolic computation consists of achieving this optimal result or getting at least tolerably close to it.

In the algebraic approach one computes a generating element p of $\overline{\mathbf{K}}(x)[y]/(f)$, the function field of $\mathcal{C}$ over the algebraic closure of $\mathbf{K}$. Then x and y can be expressed as rational functions in p, yielding a parametrization.

2. A geometric approach

Let a plane algebraic curve $\mathcal{C}$ be given by a defining polynomial, e.g. $\mathcal{C}_1$ defined by

$$f_1(x,y) = x^3 + x^2 - y^2 = 0.$$

(See Fig. 1.) $\mathcal{C}_1$ has the nice property that it contains a singular point (the origin) of multiplicity $d-1$, where d is the degree of $\mathcal{C}_1$. So if we let a line $\mathcal{L} : y = tx$ of slope t pass through the origin, then this line will intersect $\mathcal{C}_1$ in exactly one more point depending on t, namely $x = t^2 - 1$. This yields the parametrization

$$x = t^2 - 1, \quad y = t^3 - t.$$

In general, a parametrizable curve, i.e. a curve of genus 0, will not have this nice property. So what we do in the general situation is to determine a linear system of curves $\mathcal{H}_{d-2}$ of degree $d-2$ (or some other suitable degree), having every r–fold singularity of $\mathcal{C}$ as a base point of multiplicity $r-1$. Now we know that the number of intersections of $\mathcal{C}$ and $\mathcal{H}_{d-2}$ must be

$$d(d-2) = \underbrace{\sum m_P(m_P - 1)}_{=(d-1)(d-2)} + (d-2).$$

If we fix $d-3$ of the remaining $d-2$ intersections, there will be exactly one free intersection between $\mathcal{C}$ and elements of $\mathcal{H}_{d-2}$, and we can proceed as with $\mathcal{C}_1$.

As an example let us consider the curve $\mathcal{C}_2$ defined by

$$f_2(x,y) = x^6 + 3x^4y^2 - 4x^2y^2 + 3x^2y^4 + y^6 = 0.$$

(See Fig. 2.) The singularities of $\mathcal{C}_2$ in the projective plane are

$$(0:0:1) \ \text{(multiplicity 4)}, \qquad (\pm i:1:0) \ \text{(multiplicity 2)}.$$

Constructing a linear system $\mathcal{H}_4$ of curves of degree 4 through these singularities with multiplicities 3,1,1, respectively, forcing $\mathcal{H}_4$ through 3 more simple points on $\mathcal{C}_2$, and then intersecting $\mathcal{C}_2$ with $\mathcal{H}_4$ yields the parametrization

$$\begin{aligned} x &= \frac{\sqrt{2}}{250} \cdot \frac{2144374784t^6 - 2104338432t^5 - 297826560t^4 + 151338240t^3 + 6914160t^2 - 2844072t + 35721}{n(t)} \\ y &= \frac{\sqrt{2}}{250} \cdot \frac{778047488t^6 - 1619994624t^5 + 804314880t^4 + 70606080t^3 - 32017680t^2 + 795096t - 5103}{n(t)} \end{aligned}$$

where

$$n(t) = 20123648t^6 - 5326848t^5 + 2467584t^4 - 366336t^3 + 81648t^2 - 5832t + 729.$$

The key algorithmic problem is how to select the additional simple points on $\mathcal{C}$ that should be fixed on $\mathcal{H}_{d-2}$ so as to keep the necessary algebraic field extension small. This question has been solved in a joint work of the author with J.R. Sendra [9]. New and still unpublished work on this topic also makes it possible to decide whether a given curve has a real parametrization and if so compute one.

3. An algebraic approach

As mentioned in the introduction, a parametrization can be computed as a generating element of the corresponding function field of the curve $\mathcal{C}$ [1], [10]. Apparently, also in this approach simple points on the curve $\mathcal{C}$ are required and they determine the necessary field extension. Probably methods introduced in [9] could be applied also in this approach.

Fig. 1 Fig. 2

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