Preprocessing for Finite Element Discretizations of Geometric Problems

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Abstract. In this paper, we use finite element method to approximate the solutions of parameter-dependent geometric problems, and investigate the possibility of using symbolic methods as a preprocessing step. The main idea of our approach is to construct suitable finite element discretizations of the nonlinear elliptic equations leading to systems of algebraic equations, which can be subsequently solved by symbolic computation within the tolerance of computer algebra software. The prolongation of the preprocessed symbolic solution can serve as a starting value for a numerical iterative method on a finer grid.

A motivation for this approach is that usual numerical iterations (e.g. via Newton-type or fixed-point iterations) may diverge if no appropriate initial values are available. Moreover, such a purely numerical approach will not find all solutions of the discretized problem if there are more than one. A final motivation for the use of symbolic methods is the fact that all discrete solutions can be obtained as functions of unknown parameters.

In this paper, we focus on a special class of partial differential equations derived from geometric problems. A main challenge in this class is the fact that the polynomial structure of the nonlinearity is not explicit in the divergence form usually used for finite element discretization. As a consequence, the discrete form would always yield some non-polynomial terms. We therefore consider two different discretizations, namely a polynomial reformulation before discretization and a direct discretization of the divergence form with polynomial approximation of the discrete system. In order to perform a detailed analysis and convergence theory of the discretization methods we investigate some model problems related to mean-curvature type equations.

Keywords. Finite element methods, symbolic computation, preprocessing, Newton iteration, multigrid, polynomial equations.

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1. Introduction

Many geometric optimization problems can be formulated via a representation of the geometric unknown as a graph or implicitely in level set form. The corresponding optimality system then yields a system of nonlinear partial differential equations with special structure, like the famous Plateau problem (minimal surfaces) [7], free boundary problems in variational form (cf. e.g. [2]), problems in mathematical imaging (cf. e.g. [1]) or, similarly, in level set approaches to shape optimization and reconstruction (cf. e.g [16, 4]). Similar problems also arise in implicit time discretizations of geometric evolution equations like the famous *mean-curvature flow*

$$\frac{\partial u}{\partial t} = \sqrt{p^2 + |\nabla u|^2} \text{ div } \left(\frac{\nabla u}{\sqrt{p^2 + |\nabla u|^2}} \right), \tag{1}$$

with $p = 1$ for a curve or surface represented as the graph of u, and $p = 0$ for the level set representation (cf. [16] for details). The appearance of purely geometric terms like the length of the surface element $Q := \sqrt{p^2 + |\nabla u|^2}$ or the mean curvature

$$H = \text{ div } \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

is a typical characteristic of such geometric partial differential equation (by which we mean a differential equation derived from a geometric problem in graph or level set form). In many of these geometric equations real parameters (e.g. regularization parameters, cf. [4], or step sizes in time discretizations) appear, and it would be desirable to gain some information on the behavior of the solutions for different parameter values.

Due to the strong nonlinearity of differential operators involved in such geometric partial differential equation, it is often not straightforward to compute numerical solutions from a finite element discretization. For the solutions of the discrete method, Newton and fixed-point methods may fail to converge, in particular for parameter-dependent problems around critical parameter values. For discrete problems containing different roots (e.g. examples in Section 2), the numerical iterative methods are not sufficient to determine all roots and maybe may converge to undesired ones. On the other hand, if one can discretize the problem into a multivariate polynomial form, well-established symbolic methods for computing all the roots are available (cf. e.g. [19, 20]).

Since the discretized operator has similar properties as the continuous mapping, one may expect that the number of its roots does not strongly depend on the generated size and unknowns to the polynomial form, and at least limited by the BKK bound with respect to their sparse structure [17]. Practically, solving a large-scale parameter-dependent polynomial equation is more expensive than parameter-free cases, no matter whether we use symbolic computation or suitable numerical approaches. It is also important to construct a relatively cheap algorithm in order to extend the grid partition limit within the tolerance of computing software. For this sake we propose to use the symbolic method as a coarse grid solver within a multigrid framework. For the preprocessing step on the coarse grid, we do each symbolic elimination step [19] by the "subresultantchain" function from computer algebra software "CASA" [13], that could be reasonably cheap for solving such sparse polynomial system derived by a finite element discretization. Based on the convergence theory related to this kind of multigrid approach [22], the finite element solution on the fine grid can strongly benefit from the coarse grid symbolic solution, if the range of parameters do not change the elliptic property of the linearization form. In previous related work (cf. e.g. [9]) such a multigrid method with symbolic preprocessing on the coarse grid has been considered already for the Plateau problem.

We shall investigate the numerical approach with preprocessed symbolic computation for a specific model problem of the form

$$QH = \sqrt{1+|\nabla u|^2}\ \mathrm{div}\ \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \varepsilon \qquad \text{in } \Omega, \tag{2}$$

subject to Dirichlet boundary conditions $u = f$ for a given sufficiently smooth function f on $\partial\Omega$, with Ω being a regular bounded domain in $\mathbb{R}^d$ ($d = 1$ or $d = 2$). Problem (2) is a geometric partial differential equation for the curve or surface determined by the graph of u, its geometric meaning is that length of a curve (or surface) element times the mean curvature are constant (a problem related to the prescribed curvature problem, cf. [6]). It seems clear for large curvatures (i.e. large ε) one cannot expect the curve or surface to be representable by a function graph, and consequently some singular bebehavior of (2) for large ε may appear, which makes the model problem challenging. Note that (2) is also close to a forward Euler time-discretization of the mean curvature flow of graphs (cf. [5])

$$\sqrt{1+|\nabla u|^2}\ \mathrm{div}\ \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \varepsilon(u - u_0), \tag{3}$$

where $\varepsilon = (\Delta t)^{-1}$ for the time step Δt and u_0 is the (given) initial value. In Section 2 we shall also consider the symbolic preprocessing applied to (3).

A major challenge in order to apply symbolic elimination methods to finite element discretization of nonlinear partial differential equations is to transfer the discrete nonlinear system into a system of algebraic equations. The obvious way of obtaining an algebraic system after discretization consists of rewriting the geometric partial differential equation as an equation with polynomial nonlinearity before discretization. This approach immediately yields algebraic equations after finite element discretization, but as we shall see below it may destroy a divergence structure in the equation so that the weak formulation involves also second derivatives. As a consequence, one might have to use very particular discrete subspaces and the convergence analysis of the discretization may require rather strong conditions. The convergence theory with respect to this kind of discretization should also be proved. We shall detail these issues in spatial dimension two.

As an alternative to the finite element discretization based on the algebraic form, one can also perform a discretization using a weak form related to the divergence structure, i.e.,

$$\int_\Omega \frac{\nabla u \cdot \nabla v + \varepsilon v}{\sqrt{1+|\nabla u|^2}}\, dx = 0, \tag{4}$$

for all suitable test functions v (following the classical approach by Johnson and Thomeé [14] for the minimal surface problem). Such a discretization does not yield an algebraic system, but as we shall show below, the discrete system can be approximated by an algebraic system after a simple perturbation. The perturbation is of higher order in terms of the discretization size and therefore will not destroy the convergence properties of the finite element method.

The paper is organized as follows: In the Section 2 we investigate the preprocessed symbolic solution of the model problem for one dimension ($d = 1$), i.e., the graph of u representing a curve, which serves to present the basic idea in a simple way. Section 3 is devoted to a discussion of the two-dimensional case, and the difficulties arising in the direct discretization of the algebraic form, for which we provide a convergence results. In Section 4 we present results of computational experiments in the two-dimensional case and illustrate the properties of the parameter-dependent discrete solution(s) obtained by symbolic computation.

2. Preprocessed Computation of Curves

In this section, we initiate the idea by investigating (2) in the one-dimensional case $\Omega = [0, 1]$, where

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{d^2u}{dx^2}\left(1+|\frac{du}{dx}|^2\right)^{-3/2}$$

represents the curvature of the curve $\Gamma = \{(x, u(x)) \mid x \in \Omega\}$, and $Q = \sqrt{1+|\frac{du}{dx}|^2}$ is the infinitesimal curve length. Using the relation for H, (2) can be rewritten as the elliptic differential equation

$$-\frac{d^2u}{dx^2} + \varepsilon(1+(\frac{du}{dx})^2) = 0 \quad \text{in } \Omega, \qquad u(0) = u(1) = 0. \tag{5}$$

For simplicity we restrict our attention to the case $f \equiv 0$, but analogous reasoning is possible for arbitrary boundary values.

The solution of (5) only exists for $-\pi < \varepsilon < \pi$ (for larger values of $|\varepsilon|$ the curves with prescribed curvature are not function graphs) and can be computed analytically in dependence of ε as

$$u(x;\varepsilon) = \frac{1}{\varepsilon}\left[\log\left(\cos\frac{\varepsilon}{2}\right) - \log\left(\cos(\varepsilon x - \frac{\varepsilon}{2})\right)\right].$$

The graph of u as a function of x and ε is illustrated in Figure 1.

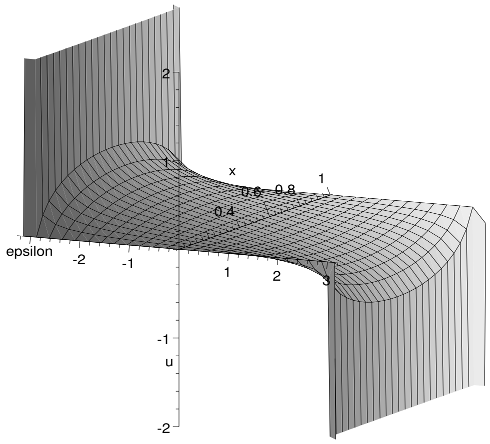


FIGURE 1. Plot of the exact solution $u(x;\varepsilon)$.

2.1. Discretization of the Algebraic Form

Let $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ be a grid on the interval $[0,1]$ and let $h = \sup_i |x_{j+1} - x_j|$. We discretize the problem (5) into piecewise linear finite elements, i.e., we define the standard basis functions φ_j, $j = 1, 2, \ldots, N$, via

$$\varphi_j(x) = \begin{cases} \frac{x-x_{j-1}}{x_j-x_{j-1}} & \text{if } x_{j-1} \le x \le x_j, \\ \frac{x_{j+1}-x}{x_{j+1}-x_j} & \text{if } x_j \le x \le x_{j+1}, \\ 0 & \text{else,} \end{cases}$$

and V^h =span$\{\varphi_j\}$. For $\varepsilon > 0$, we call $u_h(.;\varepsilon) \in V^h$ a discrete solution of (5) if

$$\int_0^1 \frac{du_h}{dx}(x;\varepsilon)\frac{dv}{dx}(x)dx + \varepsilon \int_0^1 (1 + \left(\frac{du_h}{dx}(x;\varepsilon)\right)^2) v(x)dx = 0,$$

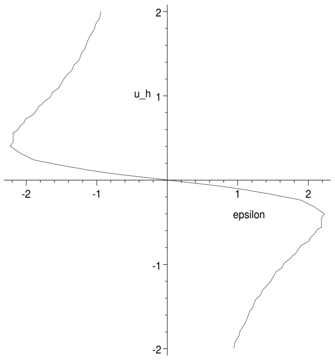
for all $v \in V^h$.

If we use the unique representation in the form

$$u_h(x;\varepsilon) = \sum_{j=1}^{N} c_j(\varepsilon)\ \varphi_j(x),$$

with a vector $\mathbf{c}(\varepsilon) = (c_j(\varepsilon))_{j=1}^N \in \mathbb{R}^N$, then the finite element discretization yields a system of quadratic equations for the vector $\mathbf{c}(\varepsilon)$. The solution of this algebraic system is then computed directly in dependence of ε via symbolic elimination approaches.

For the parameter-dependent discrete solution, we computed on the grid $x_j = \frac{j}{N+1}$, $j = 1, ..N$ respectively. In all computational experiments we observed that indeed there exists no real solutions for ε large, in particular $|\varepsilon| > \pi$, and that the finite element solution is unique for $\varepsilon = 0$. For intermediate values of $|\varepsilon|$ however, we found two different real solutions of the discrete problem, with a lower branch approximating the real solution and an upper branch diverging as $\varepsilon \to 0$. The behavior is illustrated in Figure 2 for $h = \frac{1}{4}$ and $h = \frac{1}{11}$, respectively, by plotting $u(x;\varepsilon)$ versus ε at $x = \frac{1}{2}$, respectively. One observes in particular that the second branch includes oscillations for the larger value of ε, which indicates its instability, but the first branch converge to the exact solution curve (the range of parameter ε also converge to $(-\pi, \pi)$ when the mesh size is refined).



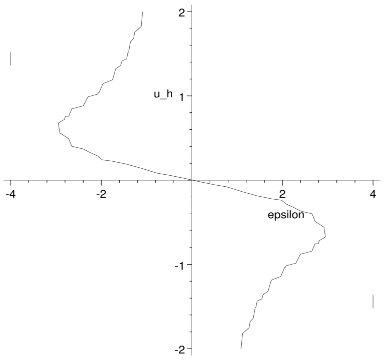


FIGURE 2. Finite element solution $u_h(\frac{1}{2};\varepsilon)$ for $h = \frac{1}{4}$ (left) and $h = \frac{1}{11}$ (right)

From our numerical results, it becomes clear that a Newton-type method might converge to the wrong discrete solution. Even if additional continuation is used, further problems might appear for ε being close to π since the convergence radius will tend to zero. However, the direct symbolic computation does not meet such critical problem and it will be simpler to compute the result close to $|\varepsilon| = \pi$. Using the symmetry property

$$u_h(\frac{j}{N+1}) = u_h(\frac{N+1-j}{N+1}), \qquad j = 1, .., N,$$

of the numerical solution one decreasing the size of the whole polynomial systems. Together with elimination taking into account by the sparsity of the polynomial form, the effort of the symbolic computation can be kept low. On a uniform grid with $h = 1/11$ (and thus 10 unknowns in the polynomial equations), the symbolic computation in the software package "CASA" only took few seconds.

Using an analogous approach, we can also compute discretize equation (3), which can be rewritten as

$$\epsilon(u-u_0)(1+\left(\frac{du}{dx}\right)^2) = \frac{d^2u}{dx^2}. \tag{6}$$

Due to the terms $u\left(\frac{du}{dx}\right)^2$ in the equation, the finite-element discretization generates a system of polynomials of degree three, and hence the complexity of the symbolic computation is higher than the previous case. However, one can still solve the coarse-grid system with reasonable effort. We illustrate the solution for $u_0 = x(1-x)$ on $\Omega = [0,1]$ by plotting $u_h(x=\frac{1}{2},.)$ as a function of the parameter ε. One observes that the behaviour is rather complicated around $\epsilon = 0$, but this would correspond to extremely large time steps in the mean-curvature flow. For the more interesting case of small time steps, there is a unique solution as one would expect from the original evolution problem.

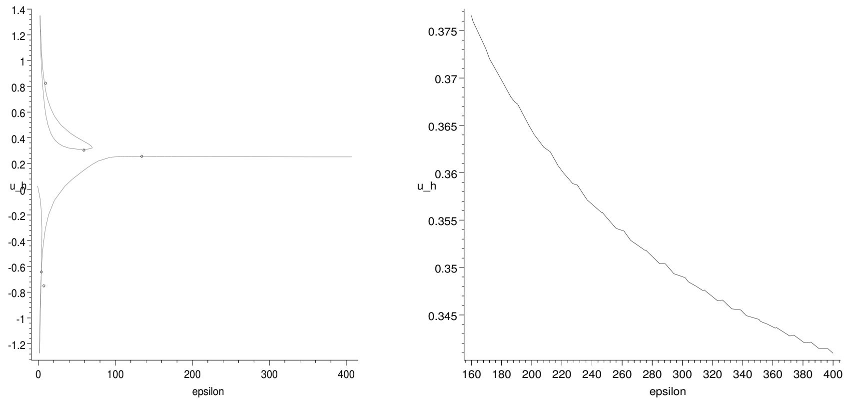


FIGURE 3. Finite element solution $u_h(\frac{1}{2};\varepsilon)$ of (6) for $h=\frac{1}{7}$

2.2. Discretization of the Divergence Form

As an alternative of a direct discretization of the algebraic form, we consider an approach via the weak form (4). We use the same finite element discretization and basis functions as introduced in the previous section, for the sake of simplicity with $x_j = \frac{j}{N+1} = jh$ (but analogous reasoning is possible for arbitrary meshes). We consequently consider the computation of a discrete solution $\tilde{u}_h(.,\varepsilon)$ satisfying

$$\int_\Omega \frac{\nabla \tilde{u}_h(.,\varepsilon)\cdot\nabla v + \varepsilon v}{\sqrt{1+|\nabla \tilde{u}_h(.,\varepsilon)|^2}}\, dx = 0, \tag{7}$$

where in this case $\nabla = \frac{d}{dx}$. Using again the representation with respect to the basis functions in the form $\tilde{u}_h(x;\varepsilon) = \sum \tilde{c}_j(\varepsilon)\varphi_j(x)$, we obtain the discrete system (with $\tilde{c}_0 = \tilde{c}_{N+1} = 0$)

$$\frac{\tilde{c}_j - \tilde{c}_{j-1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j-1})^2}} + \frac{\tilde{c}_j - \tilde{c}_{j+1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j+1})^2}} = 0,$$

$j = 1, \ldots, N$. We now multiply the equations by $h^{-2}Q_j^3$ where

$$Q_j = \sqrt{h^2 + \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j-1})^2 + \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j+1})^2},$$

to obtain the nonlinear system

$$A_j(\tilde{\mathbf{c}}) \quad = h^{-2}Q_j^3\left(\frac{\tilde{c}_j - \tilde{c}_{j-1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j-1})^2}} + \frac{\tilde{c}_j - \tilde{c}_{j+1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j+1})^2}}\right) = 0.$$

In order to obtain an approximating algebraic system, we use a Taylor expansion. More precisely, let $t_j := \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j-1})^2 - \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j+1})^2$, then

$$\begin{aligned} \frac{h^{-2}Q_j^3}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j-1})^2}} &= \frac{h^{-2}Q_j^3}{\sqrt{Q_j^2 + t_j}} = Q_j^2 - \tfrac{1}{2}h^{-2}t_j + O(t_j^2), \\ \frac{h^{-2}Q_j^3}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j+1})^2}} &= \frac{h^{-2}Q_j^3}{\sqrt{Q_j^2 - t_j}} = Q_j^2 + \tfrac{1}{2}h^{-2}t_j + O(t_j^2). \end{aligned}$$

Under sufficient smoothness assumptions we have at least $t_j = O(h^2)$, and hence the perturbation is of order h^4. Ignoring higher-order terms with respect to t_j, we obtain the approximate equation operator

$$\begin{aligned} B_j(\mathbf{c}) \quad &= (c_j - c_{j-1} + \tfrac{\varepsilon}{2}h^2)(1 + \tfrac{3}{4}h^{-2}(c_j - c_{j+1})^2 + \tfrac{1}{2}h^{-2}(c_j - c_{j-1})^2) \\ &+ (c_j - c_{j+1} + \tfrac{\varepsilon}{2}h^2)(1 + \tfrac{3}{4}h^{-2}(c_j - c_{j-1})^2 + \tfrac{1}{2}h^{-2}(c_j - c_{j+1})^2) \\ &= (2c_j - c_{j-1} - c_{j+1}) + \varepsilon(h^2 + \tfrac{1}{2}(c_j - c_{j+1})^2 + \tfrac{1}{2}(c_j - c_{j-1})^2) \\ &+ \tfrac{1}{4}h^{-2}(2c_j - c_{j-1} - c_{j+1})^3. \end{aligned}$$

We subsequently compute the discrete solution as

$$u_h(x;\varepsilon) = \sum c_j(\varepsilon)\varphi_j(x)$$

with $\mathbf{c}(\varepsilon)$ solving

$$B_j(\mathbf{c}(\varepsilon)) = 0, \qquad j = 1, \ldots, N.$$

Note that the discrete operator $\tilde{B}$ corresponding to the discretization of the algebraic form in the previous section is given by

$$\tilde{B}_j(\mathbf{c}(\varepsilon)) = (2c_j - c_{j-1} - c_{j+1}) + \varepsilon(h^2 + \frac{1}{2}(c_j - c_{j+1})^2 + \frac{1}{2}(c_j - c_{j-1})^2)$$

so that the discrete equations only differ by the higher-order term $\frac{1}{4}h^{-2}(2c_j - c_{j-1} - c_{j+1})^3$. Consequently, we may expect similar behaviour of the discrete solutions, which is indeed confirmed by the numerical experiments. In particular, we obtain again two branches of solutions, as illustrated for $h = \frac{1}{4}$ and $h = \frac{1}{11}$ in Figure 4. To compute these symbolic result is still not expensive since the polynomial form are generated by low order expansion, but we could notice the convergence speed is almost the same as in Figure 2.

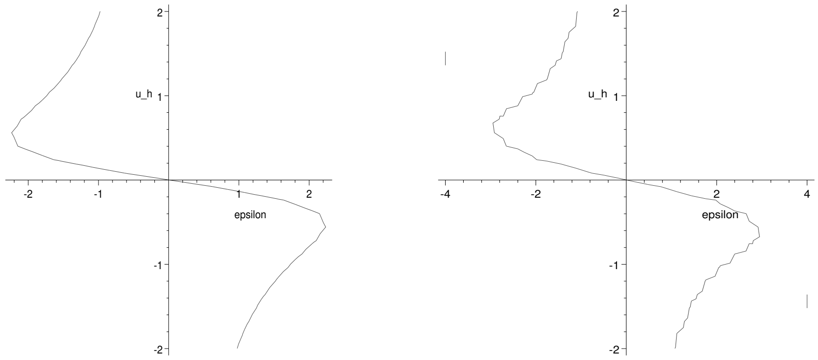


FIGURE 4. Finite solution $u_h(\frac{1}{2};\varepsilon)$ of (5) obtained from the divergence form, for $h=\frac{1}{4}$ (left) and $h=\frac{1}{11}$ (right)

3. Approximation of Surface Problems

We now consider problem (2) for $\Omega \subset \mathbb{R}^2$. With the notation $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial^2 u}{\partial xy} = u_{xy}$, we can rewrite (2) in form as

$$\text{div } ((1+u_y^2)u_x, (1+u_x^2)u_y) - 6u_{xy}u_xu_y + \varepsilon(1+u_x^2+u_y^2) = 0 \tag{8}$$

with the boundary value $u = f$ on $\partial\Omega$, where f is smooth function and Ω is a given regular domain.

Multiplying with a smooth test function v and integrating the divergence term by part, we obtain a weak formulation, namely to find u with $u = f$ on $\partial\Omega$ such that

$$\int_\Omega [(1+u_y^2)u_xv_x + (1+u_x^2)u_yv_y + 6u_{xy}u_xu_yv - \varepsilon(1+u_x^2+u_y^2)v] = 0, \tag{9}$$

for all sufficiently smooth functions v vanishing on $\partial\Omega$. Using a suitable finite element space $S_h(\Omega)$, the discretization consists in finding $u_h \in S_h(\Omega)$ with $u_h|_{\partial\Omega} = f_h$ satisfying

$$\begin{aligned}\int_\Omega &[(1+u_{h,y}^2)u_{h,x}v_x + (1+u_{h,x}^2)u_{h,y}v_y + 6u_{h,xy}u_{h,x}u_{h,y}v \\ &-\varepsilon(1+u_{h,x}^2+u_{h,y}^2)v] = 0, \ \forall\ v \in S_h^0(\Omega),\end{aligned} \tag{10}$$

where $S_h^0(\Omega)$ is the subspace of functions in $S_h(\Omega)$ vanishing on the boundary. The discretization then corresponds to an algebraic system for the coefficients with respect to a suitable set of basis functions, and this algebraic system is then solved by symbolic methods, which are introduced for the case $\varepsilon = 0$ in [8, 10].

A major difficulty in this approach is that the weak form still includes a mixed second derivative, which cannot be eliminated. In particular, in order to obtain a well-defined discrete form, the finite element space $S_h(\Omega)$ must admit mixed second derivatives, i.e.,

$$S_h(\Omega) \subset \{u \in W^{1,\infty}(\Omega) \mid u_{xy} \in L^2(\Omega)\}.$$

This property is satisfied e.g. for $S_h(\Omega)$ being a subspace of piecewise bilinear functions on a rectangular type mesh, where the edges are parallel to the coordinate axis, or for standard C^2 elements on arbitrary meshes and domains. In particular the finite element convergence analysis becomes rather complicated in such cases, as we shall work out in detail in the case of bilinear elements on a rectangular mesh.

The idea of using symbolic computation on a coarse grid has been used for the Plateau problem without parameter-dependence (cf. [9, 8]), and integrated into a two-grid algorithms with purely numerical computations on the fine grid (cf. [21, 22]). We shall discuss the solution by multigrid methods

3.1. Convergence of the Discretization

To prove the convergence theory associated to the finite element approximation (10), we first need the following lemma:

Lemma 3.1. *Let $u \in W^{2,\infty}(\Omega)$ satisfy (8), and let for $v, w \in S_h(\Omega)$,*

$$R(u,w,v) := A(w,v) - A(u,v) - A'(u;w-u,v),$$

where

$$A'(u;w,v) = \int_\Omega (\sum_{i,j=1}^{2} a_{ij} w_{(i)} v_{(j)} + \sum_{i=1}^{2} b_i w_{(i)} v)$$

is the Jacobian of (8), with the coefficients $a_{ij}, b_i, c,\ i,j = 1,2$ given by

$$a_{11}(u) = 1 + u_y^2,\ a_{22}(u) = 1 + u_x^2,\ a_{12}(u) = a_{21}(u) = -u_y u_x,$$

$$b_1(u) = 3u_y u_{xy} - 3u_{yy} u_x - 2\varepsilon u_x,$$

and

$$b_2(u) = 3u_x u_{xy} - 3u_{xx} u_y - 2\varepsilon u_y.$$

Then $u_h \in S_h(\Omega)$ solves (8) if and only if

$$A'(u;u_h,v) = R(u,u_h,v), \qquad \forall v \in S_h(\Omega).$$

Moreover, if

$$\|\partial_{xy} u_h\|_{0,\infty} + \|u_h\|_{1,\infty} \le K$$

and $S_h\Omega) = S_h^1\Omega)$ is constructed by bilinear rectangular mesh interpolation, then the remainder R satisfies

$$R(u,u_h,v) \le C(K)\|u-u_h\|_{1,\infty}^2 \|v\|_{1,1}$$

with $C(K)$ independent of ε.

Proof. Set $G(t) = A(u + t(u_h - u), v)$. Then we have identity

$$G(1) = G(0) + G_t(0) + \int_0^1 G_{tt}(t)(1-t)dt.$$

By applying identity

$$\begin{aligned}
&\int_\Omega (w_{xy} - w_{h,xy})u_x u_y v \\
&= -\int_\Omega (w_x - w_{h,x})u_{xy}u_y v + (w_x - w_{h,x})u_x u_{yy} v + (w_x - w_{h,x})u_x u_y v_y \\
&= -\int_\Omega (w_y - w_{h,y})u_{xx}u_y v + (w_y - w_{h,y})u_x u_{xy} v + (w_y - w_{h,y})u_x u_y v_x,
\end{aligned}$$

then

$$\begin{aligned}
&G_t(0) = \partial_t|_{t=0} A(u + t(u_h - u), v) \\
&= \int_\Omega [(1 + u_y^2)(u_{h,x} - u_x)v_x + (1 + u_x^2)(u_{h,y} - u_y)v_y + 2u_x u_y (u_{h,x} - u_x)v_y \\
&\ +2u_x u_y (u_{h,y} - u_y)v_x + 6u_{xy}u_y(u_{h,x} - u_x)v + 6u_{xy}u_x(u_{h,y} - u_y)v \\
&\ +6u_x u_y (u_{h,xy} - u_{yx})v - 2\varepsilon u_x(u_{h,x} - u_x)v - 2\varepsilon u_y(u_{h,y} - u_y)v] \\
&= A'(u; w_h - w, v),
\end{aligned}$$

where the coefficients of A' satisfy the given conditions of the lemma.

And, by taking

$$R(u, u_h, v) := \int_0^1 G_{tt}(t)(1-t)dt,$$

a standard estimate using the Hölder inequality yields

$$\begin{aligned}
|R(u, u_h, v)| &\le \max |G_{tt}(t)| \\
&\le C(K)\|u - u_h\|_{1,\infty}^2 \|v\|_{1,1}.
\end{aligned}$$

Finally, if u_h solves (8), then $G(1) = 0$ and this completes the proof. □

We mention that the above estimate can be derived in an analogous way for any finite element discretizations $S_h(\Omega)$ if $\|u_h\|_{2,\infty} \le K$. The convergence theory can now be derived using the fact that $A'(u; \cdot, \cdot)$ is elliptic for $\varepsilon \ne 0$.

Theorem 3.1. *Let $\Omega = [0,1]^2$, $u \in W^{2,\infty}(\Omega)$ be a solution of (8) with $u = f$ on $\partial\Omega$. Moreover, let $S_h(\Omega)$ be a finite element subspace consisting of piecewise bilinear continuous functions on a rectangular grid. Moreover, let f_h be a linear interpolation f on the boundary segments of the grid. Then there exists a constant $c > 0$, such that for h sufficiently small, there exists a solution $u_h \in S_h(\Omega)$ of (10) with $u_h = f_h$ satisfying*

$$\|u_h - u\|_{1,\infty} \le c\ h. \tag{11}$$

Proof. For any solution u, we define a nonlinear operator $\Phi : S_h(\Omega) \to S_h(\Omega)$ via

$$A'(u; \Phi(v) - u, \phi) = R(u, v, \phi), \qquad \forall \phi \in S_h(\Omega). \tag{12}$$

Using ellipticity of A', it can be shown by standard arguments that Φ is well-defined and a continuous operator. Let P_h be the standard Galerkin projection operator associated to the bilinear form A', i.e., $P_h(u) = f_h$ on $\partial\Omega$ and

$$A'(u; P_h(u) - u, \phi) = 0, \ \forall \phi \in S_h^0(\Omega).$$

For $u \in W^{2,\infty}(\Omega)$, the assumptions of Theorem 8.1.11 and Corollary 8.1.12 in [3] are satisfied, which implies the existence of a positive constant C such that

$$\|u - P_h(u)\|_{1,\infty} \leq C \ h \ \|u\|_{2,\infty}.$$

Define the set

$$B = \{v \in S_h(\Omega) : \|v - P_h(u)\|_{1,\infty} \leq Ch \ \},$$

then by inverse estimates, there exist $C_1, C_2 > 0$ depending on u only, such that

$$\begin{aligned} \|\partial_{xy}(v-u)\|_{0,\infty} &\leq \|\partial_{xy}(v - P_h(u))\|_{0,\infty} + \|\partial_{xy}(P_h(u) - u)\|_{0,\infty} \\ &\leq C_1 h^{-1} \|v - P_h(u)\|_{1,\infty} + C_2 \\ &\leq 2C_1 C + C_2. \end{aligned}$$

Hence, $\|\partial_{xy} v\|_{0,\infty}$ is uniformly bounded with respect to h for v in the subspace $S_h(\Omega)$ of bilinear elements.

We now prove that $\Phi(B) \subset B$. In fact, when We substitute ϕ in (12) by using discrete Green functions $\phi = g^z_{h,x}$ and $\phi = g^z_{h,y}$, where

$$\begin{aligned} A'(u; v, g^z_{h,x}) &= v_x(z), \\ A'(u; v, g^z_{h,y}) &= v_y(z), \end{aligned}$$

apply the definition of Galerkin projection operator P_h, the last inequality of Lemma 3.1 and the properties of discrete Green functions (cf. [18]), then we obtain for all $v \in B$,

$$\begin{aligned} \|\Phi(v) - P_h(u)\|_{1,\infty} &\leq C_0 |\log h| \|u - v\|^2_{1,\infty} \\ &\leq 2C_0 |\log h| (\|P_h(u) - v\|^2_{1,\infty} + \|u - P_h(u)\|^2_{1,\infty}) \\ &\leq 2C_0 |\log h| (C^2 h^2 + C^2 h^2) \\ &= 4C_0 C^2 |\log h| h^2 (\leq Ch), \end{aligned}$$

for h sufficiently small.

By Brouwer's fixed point theorem, there exist a solution $u_h \in B$, such that $\Phi(u_h) = u_h$. And according to Lemma 3.1, u_h solves (10) and satisfies

$$\|u_h - u\|_{1,\infty} \leq \|u_h - P_h(u)\|_{1,\infty} + \|u - P_h(u)\|_{1,\infty} \leq 2Ch.$$

□

3.2. Discretization of the Divergence Form

In the following we discuss a two-dimensional version of the approximation of the divergence form (4), which can be performed on arbitrary triangular grids. For this sake we choose a standard finite element subspace V^h consisting of continuous piecewise linear functions on a triangular grid of size h. A finite element discretization consists in finding $\tilde{u}_h \in V_h$ satisfying.

$$\int_\Omega \frac{\nabla \tilde{u}_h \cdot \nabla v + \varepsilon v}{\sqrt{1 + |\nabla \tilde{u}_h|^2}}\, dx = 0, \qquad \forall\, v \in V_h, \tag{13}$$

With the standard set of nodal basis functions φ_j satisfying $\varphi_j(x_i) = \delta_{ij}$ for all grid points x_i we can represent the discrete solution as

$$\tilde{u}(x) = \sum_{j=1}^{N} \tilde{c}_j \varphi_j(x).$$

Using the local support of the basis functions and the fact that $\nabla \tilde{u}_j$ is constant on each triangle, the finite element approximation (13) can equivalently be written as

$$Q_j^3 \sum_{T \in T(x_j)} \frac{1}{\sqrt{1 + |\nabla \tilde{u}_h|_T|^2}} \int_T (\nabla \tilde{u}_h \cdot \nabla \varphi_j + \varepsilon \varphi_j)\, dx = 0,$$

$j = 1, \ldots, N$ with an arbitrary positive factor Q_j^3, where $T(x_i)$ is the set of triangles whose nodes include x_j. Now let $A_j = \sum_{T \in T(x_j)} |T|$ be the area of the triangles surrounding x_j, then we define

$$Q_j := \sqrt{1 + \sum_{T \in T(x_j)} \frac{|T|}{A_j} |\nabla \tilde{u}_h|_T|^2}.$$

Then we can derive a similar first-order Taylor expansion of

$$\frac{Q_j^3}{\sqrt{1 + |\nabla \tilde{u}_h|_T|^2}} = \frac{Q_j^3}{\sqrt{Q_j^2 + t_j(T)}}$$

with respect to

$$\begin{aligned} t_j(T) &= |\nabla \tilde{u}_h|_T|^2 - \textstyle\sum_{T' \in T(x_j)} \frac{|T'|}{A_j} |\nabla \tilde{u}_h|_{T'}|^2 \\ &= \textstyle\sum_{T' \in T(x_j)} \frac{|T'|}{A_j} \left(|\nabla \tilde{u}_h|_T|^2 - |\nabla \tilde{u}_h|_{T'}|^2 \right). \end{aligned}$$

By similar reasoning as in the one-dimensional case we can derive an approximating weak form this way, which is polynomial (of order three) in the coefficients c_j. The arising system of algebraic system for the coefficients can subsequently be solved by symbolic elimination steps.

3.3. Multigrid Versions

At the current speed of symbolic elimination methods, the symbolic solution of the discretized problem can be carried out with reasonable efficiency by limited grid partition only. Therefore it seems natural to couple the symbolic solution technique with a multigrid approach (cf. [12] for an overview of multigrid methods), where symbolic solutions are computed on the coarse grid, and purely numerical techniques are used on finer grids.

Since the step between finer grids with numerical techniques is standard, we only discuss a two grid version, with a coarse grid (of size H) and a fine grid (of size h). In practical, the coarse finite element fomulation should be constructed reasonably within the tolerance of symbolic computing software. We assume to know suitable prolongation and interpolation operators $P_h^H : S_H(\Omega) \to S_h(\Omega)$ and $I_H^h : S_h(\Omega) \to S_H(\Omega)$, respectively. For given w, let $A'(w;\cdot,\cdot)$ be the bilinear map from Lemma 3.1. Then, we can immediately derive a cascadic multigrid method (similar to [9] for the parameter-free case) with exact symbolic solution on the coarse grid and a Newton-type correction at the fine grid which leads to higher accuracy [22, 8].

Cascadic Two-Grid Algorithm.

1. Compute $u_H \in S_H(\Omega)$ such that
$$A(u_H, v) = 0, \ \forall v \in S_H(\Omega).$$
2. Set $u_h^0 = P_h^H u_H$, and for $k = 1, \dots, k_*$ compute $u_h^k \in S_h(\Omega)$ from the linear equation
$$A'(u_h^{k-1}; u_h^k - u_h^{k-1}, v) = -A(u_h^{k-1}, v), \ \forall v \in S_h(\Omega).$$
3. Set $u_h = u_h^{k_*}$.

We finally mention that in an analogous way, standard V-cycle multigrid methods can be constructed, using the symbolic solver as a coarse grid correction, taking advantage of the fact that the exact coarse grid can also be computed in dependence of multiple parameters. Hence, if these parameters represent the values of the restriction of the fine grid solution, the symbolic coarse-grid solution can be computed in a preprocessing step, and during the V-cycle one only has to evaluate the coarse grid solution without extra computational effort.

4. Preprocessed Results for Surface Problems

In this section, we present the solutions of (10) obtained by the symbolic computation based on the computer algebra software "CASA" [13].

4.1. Discrete Solutions for Fixed Parameters

Set the boundary condition $f = 0$ and let the finite element space consists of piecewise-bilinear functions on a regular rectangular grid in the domain $\Omega =$

$[0,1] \times [0,1]$. We start with a grid consisting of 5×4 nodes, which produces (after elimination of the boundary nodes) 6 unknowns.

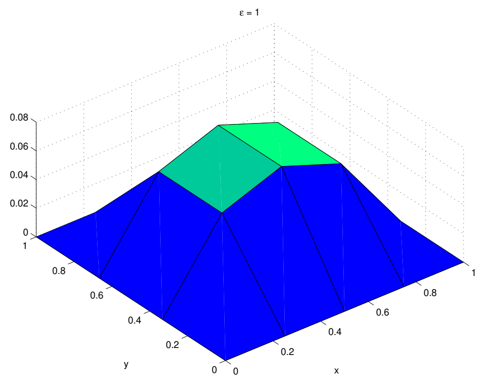


FIGURE 5. Finite element solution $u_h(\cdot;1)$.

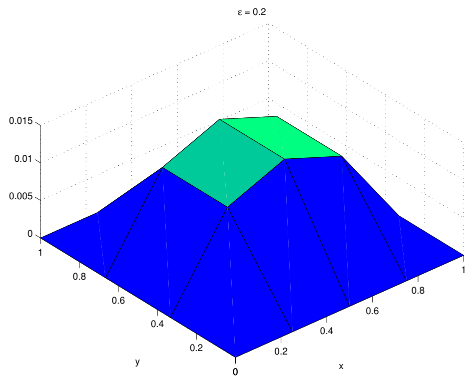


FIGURE 6. Finite element solution $u_h(\cdot;1/5)$.

We start by computing discrete solutions for fixed parameter ε on this mesh. The discrete solution turned out to be unique in all numerical experiments, which is a significant difference to the one-dimensional example, where the discrete solution was not unique or non-existent for a large range of the parameter values ε.. We illustrate the results for $\varepsilon = 1$ and $\varepsilon = 1/5$ in Figures 5 and 6.

If we change the average mesh to 5×5 nodes, we obtain a discretization fineness $h = 0.2$ and, after elimination of boundary nodes, 9 unknowns. The corresponding discrete solution for $\varepsilon = 1/5$ is plotted in Figure 7.

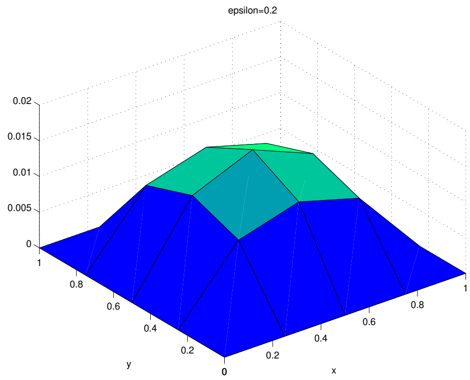


FIGURE 7. Finite element solution $u_h(\cdot; 1/5)$.

4.2. Parameter-Dependent Discrete Solutions

However, we are actually solving the elliptic equations which contains parameters. The techniques do not make any difference if we use the symboilc eliminations on a parameter-dependent discrete system. In this case, one may consider u_h as a function of the parameter ε, a relation that can be illustrated by implicit function graphs at any point $(x, y) \in \Omega$. As an result, Figure 8 shows the function graph of parameter-dependent solution $u_h(x = 0.5, y = 2/3; \varepsilon)$ based on a rectangular grid with 5×4 nodes on $\Omega = [0, 1]^2$.

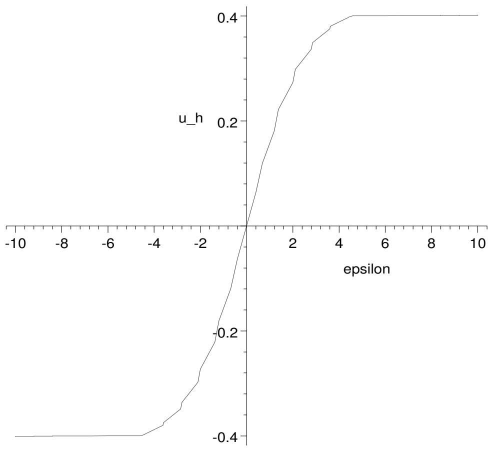


FIGURE 8. function curve of $u_h(0.5, 2/3; \varepsilon)$

From a algebraic geometry point of view, the implicit function curve dependent on ε is isomorphic, and one observes that such elliptic problems in a two-dimensional domain contains a unique solution which does not depend on parameter ε. In other cases (such as in one dimension), where the existence or uniqueness of solutions might be changed with respect to the value of ε, hysteresis would be clearly reflected by singular points in those curves.

As a further remark of this section and Section 2, due to the complexity limit of using symbolic computing software, the accuracy of the obtained experimental results are limited by the size of the domain partition. However, these preprocessed results can be very close to the convergence radius associated to the Newton type methods according to the error estimates on the coarse grid. The preprocessing techiques can not only used to save the further Newton steps (which is not trival for the parameter-dependent case) but also help to determin the number of discrete solutions. The local accuracy can also be promoted efficiently by few numerical recorrection steps, like the multigrid approach or further iterative Newton step. The typical example shows the efficiency of approximating minimal surfaces by 2-grid agorithm can be seen in [9].

5. Conclusion and Further Remarks

This paper discusses a way of solving parameter dependent elliptic equations by finite element methods and preprocessing with symbolic computation, highlighting difficulties that are obtained when discretizing geometric partial differential equations into an algebraic form. A related method has also been considered for the regularization of certain ill-posed problems in [11] (where the parameter is a regularization parameter), and can be generalized to various classes of problems, whose nonlinearity can be rewritten into (or approximated by) a polynomial form.

The size of the coarse grid problem is still strongly limited, even if the symbolic approach took the advantage of the sparsity structure in the discrete problem. However, the preprocessing helps to obtain numerical solutions on fine grids with lower effort and higher reliability.

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