MACMAHON'S PARTITION ANALYSIS IX: k-GON PARTITIONS

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Dedicated to George Szekeres on the occasion of his 90th birthday

ABSTRACT. MacMahon devoted a significant portion of Volume II of his famous book "Combinatory Analysis" to the introduction of Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations. In a series of papers we have shown that MacMahon's method turns into an extremely powerful tool when implemented in computer algebra. In this note we explain how the use of the package `Omega` developed by the authors has led to a generalization of a classical counting problem related to triangles with sides of integer length.

1. INTRODUCTION

In his famous book "Combinatory Analysis" [12, Vol. II, Sect. VIII, pp. 91–170] MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations.

We will use MacMahon's method and the `Omega` package for a study of a classical combinatorial problem related to triangles with sides of integer size. We start out by stating the well-known base case which has been discussed at various places; see e.g. [11, 9, 1, 10, 8], and [13, Ch. 4, Ex. 16].

Problem 1. Let $t_3(n)$ be the number of non-congruent triangles whose sides have integer length and whose perimeter is n. For instance, $t_3(9) = 3$, corresponding to $3+3+3$, $2+3+4$, $1+4+4$. Find $\sum_{n\geq 3} t_3(n)q^n$.

Obviously the corresponding generating function is

$$T_3(q) := \sum_{n\geq 3} t_3(n)\, q^n = \sum\nolimits^{*} q^{a_1+a_2+a_3} \tag{1}$$

where $\sum^{*}$ is the restricted summation over all positive integer triples (a_1, a_2, a_3) satisfying $a_1 \leq a_2 \leq a_3$ and $a_1 + a_2 > a_3$.

In order to see how Partition Analysis can be used to compute a closed form representation for $\sum_{n\geq 3} t_3(n)q^n$, we need to recall the key ingredient of MacMahon's method, the Omega operator $\Omega_{\geqq}$.

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Definition 1. The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\ldots,s_r}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of $A_{s_1,\ldots,s_r}$.

We emphasize that it is essential to treat everything analytically rather than formally because the method relies on unique Laurent series representations of rational functions.

Another fundamental aspect of Partition Analysis is the use of elimination rules which describe the action of the Omega operator on certain base cases. MacMahon begins the discussion of his method by presenting a catalog [12, Vol. II, pp. 102–106] of fundamental evaluations. Subsequently he extends this table by new rules whenever he is enforced to do so. Once found, most of these fundamental rules are easy to prove. This is illustrated by the following examples which are taken from MacMahon [12, Vol. II, Art. 354, p. 106].

Proposition 1. *For integer $s \geq 0$ and variables A, B being free of λ,*

$$\underset{\geqq}{\Omega} \frac{\lambda^{-s}}{(1-\lambda A)\left(1-\frac{B}{\lambda}\right)} = \frac{A^s}{(1-A)(1-AB)}; \tag{2}$$

$$\underset{\geqq}{\Omega} \frac{\lambda^{s}}{(1-\lambda A)\left(1-\frac{B}{\lambda}\right)} = \frac{1-AB-B^{s+1}+AB^{s+1}}{(1-A)(1-B)(1-AB)}. \tag{3}$$

Proof. Rule (3) is a special case of the more general rule (13); see Lemma 2. Rule (2) is proved as follows. By geometric series expansion the left-hand side equals

$$\underset{\geqq}{\Omega} \sum_{i,j\geq 0} \lambda^{i-j-s} A^i B^j = \underset{\geqq}{\Omega} \sum_{j,k\geq 0} \lambda^k A^{k+j+s} B^j,$$

where the summation parameter i has then been replaced by $k + j + s$. But now $\Omega_{\geqq}$ sets λ to 1, which completes the proof. □

Now we are ready for deriving the closed form expression for $T_3(q)$ with Partition Analysis.

First, in order to get rid of the diophantine constraints, one rewrites the restricted sum expression in (1) into what MacMahon has called the "crude form" of the generating function,

$$\begin{aligned} T_3(q) &= \underset{\geqq}{\Omega} \sum_{a_1\geq 1,\ a_2,a_3\geq 0} \lambda_1^{a_2-a_1} \lambda_2^{a_3-a_2} \lambda_3^{a_1+a_2-a_3-1} q^{a_1+a_2+a_3} \\ &= \underset{\geqq}{\Omega} \frac{q\lambda_1^{-1}}{\left(1-q\frac{\lambda_3}{\lambda_1}\right)\left(1-q\frac{\lambda_1\lambda_3}{\lambda_2}\right)\left(1-q\frac{\lambda_2}{\lambda_3}\right)}, \end{aligned}$$

where the last line is by geometric series summation.

Next by applying again rule (2) we eliminate successively λ_2, λ_1, and λ_3,

$$\begin{aligned} T_3(q) &= \underset{\geq}{\Omega} \frac{q\lambda_1^{-1}}{\left(1 - q\frac{\lambda_3}{\lambda_1}\right)\left(1 - \frac{q}{\lambda_3}\right)(1 - q^2\lambda_1)} \\ &= \underset{\geq}{\Omega} \frac{q^3}{(1-q^2)(1-q^3\lambda_3)\left(1 - \frac{q}{\lambda_3}\right)} \\ &= \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}. \end{aligned} \tag{4}$$

This completes the generating function computation and Problem 1 is solved.

With our package `Omega`[1] the whole computation can be done automatically and in one stroke. Note that setting-up the crude generating function is done also by the package:

```
In[1]:= <<Omega2.m
Out[1]= Axel Riese's Omega implementation version 2.33 loaded
In[2]:= OSum[q^(a1+a2+a3), {a2 ≥ a1, a3 ≥ a2, a1+a2 > a3, a1 ≥ 1}, λ]
        Assuming a2 ≥ 0
        Assuming a3 ≥ 0
Out[2]=
```

$$\underset{\lambda_1,\lambda_2,\lambda_3}{\underset{\geq}{\Omega}} \frac{q}{\lambda_1\left(1 - \frac{q\,\lambda_2}{\lambda_3}\right)\left(1 - \frac{q\,\lambda_3}{\lambda_1}\right)\left(1 - \frac{q\,\lambda_1\,\lambda_3}{\lambda_2}\right)}$$

```
In[3]:= OR[%]
        Eliminating λ3...
        Eliminating λ2...
        Eliminating λ1...
Out[4]=
```

$$\frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}$$

As already pointed out in [2], with Partition Analysis one is able to derive much more information. Namely, we can consider the full generating function

$$S_3(x_1, x_2, x_3) := \sum\nolimits^* x_1^{a_1} x_2^{a_2} x_3^{a_3},$$

where $\sum^*$ denotes again the restricted summation over all positive integer triples (a_1, a_2, a_3) satisfying $a_1 \leq a_2 \leq a_3$ and $a_1 + a_2 > a_3$. On this expression we can carry out essentially the same Partition Analysis steps as above for obtaining a closed form expression for it. For the crude form one gets

$$\begin{aligned} S_3(x_1, x_2, x_3) &= \underset{\geq}{\Omega} \sum_{a_1 \geq 1,\ a_2, a_3 \geq 0} \lambda_1^{a_2 - a_1} \lambda_2^{a_3 - a_2} \lambda_3^{a_1 + a_2 - a_3 - 1} x_1^{a_1} x_2^{a_2} x_3^{a_3} \\ &= \underset{\geq}{\Omega} \frac{x_1 \lambda_1^{-1}}{\left(1 - x_1\frac{\lambda_3}{\lambda_1}\right)\left(1 - x_2\frac{\lambda_1\lambda_3}{\lambda_2}\right)\left(1 - x_3\frac{\lambda_2}{\lambda_3}\right)}. \end{aligned}$$

[1]available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega*

Next by applying again rule (2), we eliminate successively λ_2, λ_1, and λ_3 as above and obtain

$$\begin{aligned}S_3(x_1,x_2,x_3) &= \underset{\geqq}{\Omega}\,\frac{x_1\lambda_1^{-1}}{\left(1-x_1\frac{\lambda_3}{\lambda_1}\right)\left(1-\frac{x_3}{\lambda_3}\right)(1-x_2x_3\lambda_1)}\\ &= \underset{\geqq}{\Omega}\,\frac{x_1x_2x_3}{(1-x_2x_3)(1-x_1x_2x_3\lambda_3)\left(1-\frac{x_3}{\lambda_3}\right)}\\ &= \frac{x_1x_2x_3}{(1-x_2x_3)(1-x_1x_2x_3)(1-x_1x_2x_3^2)}. \qquad (5)\end{aligned}$$

This not only generalizes the generating function $T_3(q)$, i.e. $T_3(q) = S_3(q,q,q)$, but gives rise also to a complete, parameterized solution of the underlying diophantine set of equations

$$1 \le a_1,\ a_1 \le a_2,\ a_2 \le a_3,\ \text{and } a_1 + a_2 > a_3.$$

This can be seen by geometric series expansion of (5), namely

$$S_3(x_1,x_2,x_3) = \sum_{n_1,n_2,n_3\ge 0} x_1^{n_2+n_3+1}x_2^{n_1+n_2+n_3+1}x_3^{n_1+n_2+2n_3+1}.$$

In other words, by choosing

$$a_1 = n_2+n_3+1, a_2 = n_1+n_2+n_3+1,\ \text{and } a_3 = n_1+n_2+2n_3+1,$$

and running through all non-negative integers n_1, n_2, n_3, one constructs in a one-to-one fashion *all* non-degenerate triangles with sides of integer size.

In [3] we considered the following generalization of the triangle problem to k-gons where $k \ge 3$.

Definition 2. As the set of *non-degenerate k-gon partitions into positive parts* we define

$$\tau_k := \{(a_1,\dots,a_k) \in \mathbb{Z}^k \mid 1 \le a_1 \le a_2 \le \cdots \le a_k \text{ and } a_1+\cdots+a_{k-1} > a_k\}.$$

As the set of *non-degenerate k-gon partitions of n into positive parts* we define

$$\tau_k(n) := \{(a_1,\dots,a_k) \in \tau_k \mid a_1+\cdots+a_k = n\}.$$

The corresponding cardinality is denoted by

$$t_k(n) := |\tau_k(n)|.$$

The term "non-degenerate" refers to the restriction to strict inequality, i.e. to $a_1+\cdots+a_{k-1} > a_k$. In the form of (4) we computed a rational expression for $T_3(q) = \sum_{n\ge 3} t_3(n)q^n$. With the `Omega` package in hand, we are able to compute also the next cases in a purely mechanical manner. For instance,

$$\sum_{n\ge 4} t_4(n)\,q^n = \frac{q^4(1+q+q^5)}{(1-q^2)(1-q^3)(1-q^4)(1-q^6)}, \qquad (6)$$

$$\sum_{n\ge 5} t_5(n)\,q^n = \frac{q^5(1-q^{11})}{(1-q)(1-q^2)(1-q^4)(1-q^5)(1-q^6)(1-q^8)}, \qquad (7)$$

and

$$\sum_{n\ge 6} t_6(n)\,q^n = \frac{q^6(1-q^4+q^5+q^7-q^8-q^{13})}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^6)(1-q^8)(1-q^{10})}. \qquad (8)$$

From these results we were able to derive a number of partition theoretical consequences. However, despite the fact that the particular instances of $\sum_{n\geq k} t_k(n)q^n$ can be computed so easily, we were not able to find a common underlying pattern. So we stated as an open problem:

Problem 2. In view of the generating function representations (4), (6), (7), and (8): Is it possible to find a common pattern for all possible choices of k?

In Section 2 we provide an affirmative answer to this problem. More precisely, we give closed form expressions for $T_k(q)$ as well as for the corresponding general version $S_k(x_1, \ldots, x_k)$ defined as follows:

Definition 3. For integer $k \geq 3$,

$$T_k(q) := \sum_{n\geq k} t_k(n)\, q^n,$$

and

$$S_k(x_1, \ldots, x_k) := \sum_{(a_1,\ldots,a_k)\in\tau_k} x_1^{a_1} \cdots x_k^{a_k}.$$

Finally, Section 3 provides some concluding remarks.

2. Generating Functions for k-Gon Partitions

In this section we prove the following main result for k-gon partitions.

Theorem 1. *Let $k \geq 3$ and $X_i = x_i \cdots x_k$ for $1 \leq i \leq k$. Then*

$$\begin{aligned} S_k(x_1, \ldots, x_k) &= \frac{X_1}{(1-X_1)(1-X_2)\cdots(1-X_k)} \\ &\quad - \frac{X_1 X_k^{k-2}}{1-X_k}\, \frac{1}{(1-X_{k-1})(1-X_{k-2}X_k)(1-X_{k-3}X_k^2)\cdots(1-X_1X_k^{k-2})}. \end{aligned} \tag{9}$$

Since $T_k(q) = S_k(q, \ldots, q)$, Theorem 1 implies the desired generating function representation.

Corollary 1. *For $k \geq 3$,*

$$T_k(q) = \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)} - \frac{q^{2k-2}}{1-q}\, \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2k-2})}. \tag{10}$$

Remark. It is easily verified that (10) is bringing the representations (4), (6), (7), and (8) of the special cases $k = 3, 4, 5, 6$ under one umbrella.

We shall prove Theorem 1 with Partition Analysis. To this end we first need the crude form of $S_k(x_1, \ldots, x_k)$.

Proposition 2. *For $k \geq 3$,*

$$\begin{aligned} &S_k(x_1, \ldots, x_k) \\ &\quad = \underset{\geq}{\Omega} \frac{x_1\lambda_1^{-1}}{\left(1 - x_1\frac{\lambda_k}{\lambda_1}\right)\left(1 - x_2\frac{\lambda_1\lambda_k}{\lambda_2}\right)\left(1 - x_3\frac{\lambda_2\lambda_k}{\lambda_3}\right)\cdots\left(1 - x_{k-1}\frac{\lambda_{k-2}\lambda_k}{\lambda_{k-1}}\right)\left(1 - x_k\frac{\lambda_{k-1}}{\lambda_k}\right)}. \end{aligned} \tag{11}$$

Proof. For fixed integer $k \geq 3$,

$$S_k(x_1, \ldots, x_k) = \underset{\geqq}{\Omega} \sum_{\substack{a_1 \geq 1 \\ a_2, \ldots, a_k \geq 0}} x_1^{a_1} \cdots x_k^{a_k} \lambda_1^{a_2 - a_1} \cdots \lambda_{k-1}^{a_k - a_{k-1}} \lambda_k^{a_1 + \cdots + a_{k-1} - a_k - 1}$$

by the definition of $\Omega_{\geqq}$. The rest follows by geometric series summation. □

The next step is the successive elimination of $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ from the crude form (11). For this it is convenient to introduce a lemma.

Lemma 1. *Let $k \geq 3$ and let $y_1, \ldots, y_k$ be free of $\lambda_1, \ldots, \lambda_{k-1}$. Then*

$$\begin{aligned} &\underset{\geqq}{\Omega} \frac{y_1 \lambda_1^{-1}}{\left(1 - \frac{y_1}{\lambda_1}\right)\left(1 - y_2 \frac{\lambda_1}{\lambda_2}\right) \cdots \left(1 - y_{k-1} \frac{\lambda_{k-2}}{\lambda_{k-1}}\right)(1 - y_k \lambda_{k-1})} \\ &\quad = \frac{y_1 \cdots y_k}{(1 - y_k)(1 - y_{k-1} y_k) \cdots (1 - y_1 \cdots y_k)}. \end{aligned}$$

Proof. We proceed by induction on k. For $k = 3$,

$$\begin{aligned} &\underset{\geqq}{\Omega} \frac{y_1 \lambda_1^{-1}}{\left(1 - \frac{y_1}{\lambda_1}\right)\left(1 - y_2 \frac{\lambda_1}{\lambda_2}\right)(1 - y_3 \lambda_2)} \\ &\quad = \underset{\geqq}{\Omega} \frac{y_1 \lambda_1^{-1}}{\left(1 - \frac{y_1}{\lambda_1}\right)(1 - y_3)(1 - y_2 y_3 \lambda_1)} && \text{(by (2) with } s = 0) \\ &\quad = \frac{y_1 y_2 y_3}{(1 - y_2 y_3)(1 - y_3)(1 - y_1 y_2 y_3)}. && \text{(by (2) with } s = 1) \end{aligned}$$

For the induction step we apply again rule (2) with $s = 0$,

$$\begin{aligned} &\underset{\geqq}{\Omega} \frac{y_1 \lambda_1^{-1}}{\left(1 - \frac{y_1}{\lambda_1}\right)\left(1 - y_2 \frac{\lambda_1}{\lambda_2}\right) \cdots \left(1 - y_{k-1} \frac{\lambda_{k-2}}{\lambda_{k-1}}\right)\left(1 - y_k \frac{\lambda_{k-1}}{\lambda_k}\right)(1 - y_{k+1} \lambda_k)} \\ &\quad = \frac{1}{1 - y_{k+1}} \underset{\geqq}{\Omega} \frac{y_1 \lambda_1^{-1}}{\left(1 - \frac{y_1}{\lambda_1}\right)\left(1 - y_2 \frac{\lambda_1}{\lambda_2}\right) \cdots \left(1 - y_{k-1} \frac{\lambda_{k-2}}{\lambda_{k-1}}\right)(1 - y_k y_{k+1} \lambda_{k-1})} \\ &\quad = \frac{1}{1 - y_{k+1}} \frac{y_1 \cdots y_{k+1}}{(1 - y_k y_{k+1})(1 - y_{k-1} y_k y_{k+1}) \cdots (1 - y_1 \cdots y_{k+1})}; \end{aligned}$$

for the last line we used the induction hypothesis. □

Now we are in the position to state the crude form of $S_k(x_1, \ldots, x_k)$.

Proposition 3. *Let $k \geq 3$ and $X_i = x_i \cdots x_k$ for $1 \leq i \leq k$. Then*

$$\begin{aligned} &S_k(x_1, \ldots, x_k) \\ &\quad = \frac{X_1}{1 - X_{k-1}} \underset{\geqq}{\Omega} \frac{\lambda_k^{k-3}}{\left(1 - \frac{X_k}{\lambda_k}\right)(1 - X_{k-2} \lambda_k)(1 - X_{k-3} \lambda_k^2) \cdots (1 - X_1 \lambda_k^{k-2})}. \end{aligned} \tag{12}$$

Proof. By Proposition 2,

$$S_k(x_1, \ldots, x_k) = \underset{\geqq}{\Omega} \frac{y_1 \lambda_1^{-1} \lambda_k}{\left(1 - \frac{y_1}{\lambda_1}\right)\left(1 - y_2 \frac{\lambda_1}{\lambda_2}\right) \cdots \left(1 - y_{k-1} \frac{\lambda_{k-2}}{\lambda_{k-1}}\right)(1 - y_k \lambda_{k-1})},$$

where $y_1 = x_1 \lambda_k, \ldots, y_{k-1} = x_{k-1} \lambda_k$ and $y_k = x_k / \lambda_k$. By Lemma 1 this is equal to

$$\underset{\geqq}{\Omega} \frac{x_1 \cdots x_k \lambda_k^{k-3}}{\left(1 - \frac{x_k}{\lambda_k}\right)(1 - x_{k-1} x_k)(1 - x_{k-2} x_{k-1} x_k \lambda_k) \cdots (1 - x_1 \cdots x_k \lambda_k^{k-2})}$$

which is the right-hand side of (12). □

In order to complete the proof of Theorem 1 we need another elementary lemma; namely, the special case $m = 1$, $k = 1$, and $j_i = i$ of our reduction algorithm described in [4]. However, for the sake of better readability we state and prove it explicitly.

Lemma 2. *Let $k \geq 1$, $a \geq 0$, and let $y, y_1, \ldots, y_k$ be free of λ. Then*

$$\underset{\geqq}{\Omega} \frac{\lambda^a}{(1-\frac{y}{\lambda})(1-y_1\lambda)(1-y_2\lambda^2)\cdots(1-y_k\lambda^k)} = \frac{1}{(1-y_1)\cdots(1-y_k)(1-y)} - \frac{y^{a+1}}{(1-y_1y)(1-y_2y^2)\cdots(1-y_ky^k)(1-y)}. \tag{13}$$

Remark. Formula (3) of Proposition 1 is the special case $k = 1$.

Proof. The left hand-side of (13) equals

$$\underset{\geqq}{\Omega} \sum_{s_1,\ldots,s_k\geq 0} \sum_{r\geq 0} y_1^{s_1}\cdots y_k^{s_k} y^r \lambda^{1\cdot s_1+2\cdot s_2+\cdots+k\cdot s_k+a-r} = \sum_{s_1,\ldots,s_k\geq 0} y_1^{s_1}\cdots y_k^{s_k} \sum_{r=0}^{1\cdot s_1+\cdots+k\cdot s_k+a} y^r$$

and the lemma follows by applying $\sum_{r=0}^{m} y^r = (1-y^{m+1})/(1-y)$. □

Finally we come to the proof of Theorem 1.

Proof of Theorem 1. By Proposition 3,

$$\begin{aligned} S_k(x_1,\ldots,x_k) &= \frac{X_1}{1-X_{k-1}} \underset{\geqq}{\Omega} \frac{\lambda_k^{k-3}}{(1-\frac{X_k}{\lambda_k})(1-X_{k-2}\lambda_k)(1-X_{k-3}\lambda_k^2)\cdots(1-X_1\lambda_k^{k-2})} \\ &= \frac{X_1}{1-X_{k-1}} \left(\frac{1}{(1-X_1)(1-X_2)\cdots(1-X_{k-2})(1-X_k)} \right. \\ &\qquad \left. - \frac{X_k^{k-2}}{(1-X_{k-2}X_k)(1-X_{k-3}X_k^2)\cdots(1-X_1X_k^{k-2})(1-X_k)} \right), \end{aligned}$$

where the last equality is by Lemma 2 with $a = k-3$ and $y = X_k, y_1 = X_{k-2}, y_2 = X_{k-3}, \ldots, y_{k-2} = X_1$. This completes the proof of Theorem 1. □

3. Conclusion

As shown in a series of articles [3, 4, 5, 6, 7], Partition Analysis is ideally suited for being supplemented by computer algebra methods. In these papers the Mathematica package `Omega` which had been developed by the authors, was used as an essential tool.

The `Omega` package played a crucial role also in discovering Theorem 1 above. However, it is important to note that the computations (4), (6), (7), and (8) for $T_k(q)$ with $k = 3, 4, 5, 6$ have not led us to Theorem 1. Rather than this, the main point in the study of k-gon partitions was the careful `Omega` investigation of the full generating function $S_k(x_1, \ldots, x_k)$; only in this generality the underlying pattern was finally revealed.

Another remark concerns the constructive use of Theorem 1. As a matter of fact, formula (9) can be used to *construct* k-gon partitions in the same way as explained in the introduction with the special case (5).

In [2] refinements of the base case $k = 3$ of Theorem 1 and Corollary 1 have been considered. We expect that experiments with the `Omega` package will lead to more general results in this direction.

References

[1] G.E. Andrews, *A note on partitions and triangles with integer sides*, Amer. Math. Monthly **86** (1979), 477–478.

[2] ———, *MacMahon's partition analysis II: Fundamental theorems*, Ann. Comb. **4** (2000).

[3] G.E. Andrews, P. Paule and A. Riese, *MacMahon's partition analysis III: The `Omega` package*, SFB Report **99-24**, J. Kepler University, Linz, 1999. (to appear)

[4] ———, *MacMahon's partition analysis VI: A new reduction algorithm*, SFB Report **01-4**, J. Kepler University, Linz, 2001. (to appear)

[5] ———, *MacMahon's partition analysis VII: Constrained compositions*, SFB Report **01-5**, J. Kepler University, Linz, 2001. (to appear)

[6] ———, *MacMahon's partition analysis VIII: Plane Partition Diamonds*, SFB Report **01-6**, J. Kepler University, Linz, 2001. (to appear)

[7] G.E. Andrews, P. Paule, A. Riese and V. Strehl, *MacMahon's partition analysis V: Bijections, recursions, and magic squares*, SFB Report **00-18**, J. Kepler University, Linz, 2000. (to appear)

[8] R. Honsberger, *Mathematical Gems III*, Math. Assoc. of America, Washington, 1985.

[9] J.H. Jordan, R. Walsh and R.J. Wisner, *Triangles with integer sides*, Notices Amer. Math. Soc. **24** (1977), A-450.

[10] J.H. Jordan, R. Walsh, and R.J. Wisner, *Triangles with integer sides*, Amer. Math. Monthly **86** (1979), 686–689.

[11] C.I. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, 1968.

[12] P.A. MacMahon, *Combinatory Analysis*, 2 vols., Cambridge University Press, Cambridge, 1915–1916. (Reprinted: Chelsea, New York, 1960)

[13] R.P. Stanley, *Enumerative Combinatorics — Volume* 1, Wadsworth, Monterey, California, 1986.

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