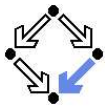
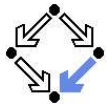
# Model Checking (Part 3)

Wolfgang Schreiner
Wolfgang.Schreiner@risc.uni-linz.ac.at

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University, Linz, Austria
http://www.risc.uni-linz.ac.at



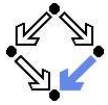








## The Basic Approach



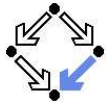
Translation of the original problem to a problem in automata theory.

- Original problem: $S \models P$.
  - $S = \langle I, R \rangle$, PLTL formula $P$.
  - Does property $P$ hold for every run of system $S$?
- Construct system automaton $S_A$ with language $\mathcal{L}(S_A)$.
  - A language is a set of infinite words.
  - Each such word describes a system run.
  - $\mathcal{L}(S_A)$ describes the set of runs of $S$.
- Construct property automaton $P_A$ with language $\mathcal{L}(P_A)$.
  - $\mathcal{L}(P_A)$ describes the set of runs satisfying $P$.
- Equivalent Problem: $\mathcal{L}(S_A) \subseteq \mathcal{L}(P_A)$.
  - The language of $S_A$ must be contained in the language of $P_A$.

There exists an efficient algorithm to solve this problem.



## Finite State Automata

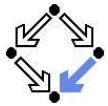


A (variant of a) labeled transition system in a finite state space.

- Take finite sets *State* and *Label*.
  - The state space *State*.
  - The alphabet *Label*.
- A (finite state) automaton $A = \langle I, R, F \rangle$ over *State* and *Label*:
  - A set of initial states $I \subseteq State$.
  - A labeled transition relation $R \subseteq Label \times State \times State$.
  - A set of final states $F \subseteq State$.
    - Büchi automata: $F$ is called the set of accepting states.

We will only consider infinite runs of Büchi automata.

## Runs and Languages



- An infinite run $r = s_0 \xrightarrow{I_0} s_1 \xrightarrow{I_1} s_2 \xrightarrow{I_2} \ldots$ of automaton $A$:
  - $s_0 \in I$ and $R(I_i, s_i, s_{i+1})$ for all $i \in \mathbb{N}$.
  - Run $r$ is said to read the infinite word $w(r) := \langle I_0, I_1, I_2, \ldots \rangle$.
- $A = \langle I, R, F \rangle$ accepts an infinite run $r$:
  - Some state $s \in F$ occurs infinitely often in $r$.
  - This notion of acceptance is also called Büchi acceptance.
- The language $\mathcal{L}(A)$ of automaton $A$:
  - $\mathcal{L}(A) := \{w(r) : A \text{ accepts } r\}$.
  - The set of words which are read by the runs accepted by $A$.
- Example: $\mathcal{L}(A) = (a^*bb^*a)^*a^\omega + (a^*bb^*a)^\omega = (b^*a)^\omega$.
  - $w^i = ww \ldots w$ ($i$ occurrences of $w$).
  - $w^* = \{w^i : i \in \mathbb{N}\} = \{\langle\rangle, w, ww, www, \ldots\}$.
  - $w^\omega = wwww \ldots$ (infinitely often).
  - An infinite repetition of an arbitrary number of $b$ followed by $a$.

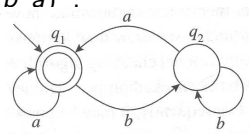
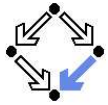



**Figure 9.1**
A finite automaton.

Edmund Clarke: "Model Checking", 1999.



## A Finite State System as an Automaton



The automaton $S_A = \langle I, R, F \rangle$ for a finite state system $S = \langle I_S, R_S \rangle$:

- $State := State_S \cup \{\iota\}$.
  - The state space $State_S$ of $S$ is finite; additional state $\iota$ ("iota").
- $Label := \mathbb{P}(AP)$.
  - Finite set $AP$ of atomic propositions.
    All PLTL formulas are built from this set only.
  - Powerset $\mathbb{P}(S) := \{s : s \subseteq S\}$.
  - Every element of $Label$ is thus a set of atomic propositions.
- $I := \{\iota\}$.
  - Single initial state $\iota$.
- $R(I, s, s') :\Leftrightarrow I = L(s') \wedge (R_S(s, s') \vee (s = \iota \wedge I_S(s')))$.
  - $L(s) := \{p \in AP : s \models p\}$.
  - Each transition is labeled by the set of atomic propositions satisfied by the successor state.
  - Thus all atomic propositions are evaluated on the successor state.
- $F := State$.
  - Every state is accepting.



## A Finite State System as an Automaton



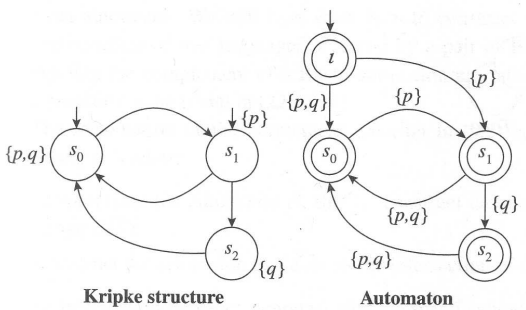


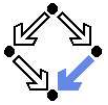

**Figure 9.2**
Transforming a Kripke structure into an automaton.

Edmund Clarke et al: "Model Checking", 1999.

If $r = s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots$ is a run of $S$, then $S_A$ accepts the labelled version $r_l := \iota \xrightarrow{L(s_0)} s_0 \xrightarrow{L(s_1)} s_1 \xrightarrow{L(s_2)} s_2 \xrightarrow{L(s_3)} \ldots$ of $r$.

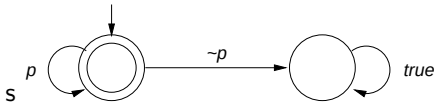


## A System Property as an Automaton

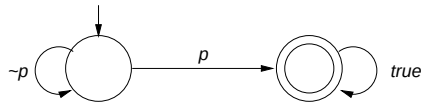


Also an PLTL formula can be translated to a finite state automaton.

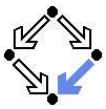
- We need the automaton $P_A$ for a PLTL property $P$.
  - Requirement: $r \models P \Leftrightarrow P_A$ accepts $r_l$.
  - A run satisfies property $P$ if and only if automaton $A_P$ accepts the labeled version of the run.
- Example: $\Box p$.



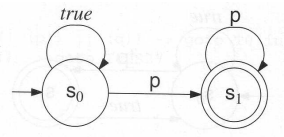

- Example: $\Diamond p$.



## Further Examples

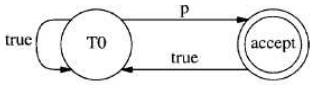


- Example: $\Diamond\Box p$.




Gerard Holzmann: "The Spin Model Checker", 2004.

- Example: $\Box\Diamond p$.

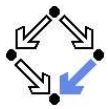



Gerard Holzmann: "The Model Checker Spin", 1997.

We will give later an algorithm to convert arbitrary PLTL formulas to automata.



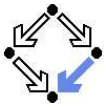
## System Properties



- State equivalence: $L(s) = L(t)$.
  - Both states have the same labels.
  - Both states satisfy the same atomic propositions in $AP$.
- Run equivalence: $w(r_l) = w(r'_l)$.
  - Both runs have the same sequences of labels.
  - Both runs satisfy the same PLTL formulas built over $AP$.
- Indistinguishability: $w(r_l) = w(r'_l) \Rightarrow (r \models P \Leftrightarrow r' \models R)$
  - PLTL formula $P$ cannot distinguish between runs $r$ and $r'$ whose labeled versions read the same words.
- Consequence: $S \models P \Leftrightarrow \mathcal{L}(S_A) \subseteq \mathcal{L}(P_A)$.
  - Proof that, if every run of $S$ satisfies $P$, then every word $w(r_l)$ in $\mathcal{L}(S_A)$ equals some word $w(r'_l)$ in $\mathcal{L}(P_A)$, and vice versa.
  - "Vice versa" direction relies on indistinguishability property.



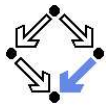
## The Next Steps



- Problem: $\mathcal{L}(S_A) \subseteq \mathcal{L}(P_A)$
  - Equivalent to: $\mathcal{L}(S_A) \cap \overline{\mathcal{L}(P_A)} = \emptyset$.
    - Complement $\overline{L} := \{w : w \notin L\}$.
  - Equivalent to: $\mathcal{L}(S_A) \cap \mathcal{L}(\neg P_A) = \emptyset$.
    - $\overline{\mathcal{L}(A)} = \mathcal{L}(\neg A)$.
- Equivalent Problem: $\mathcal{L}(S_A) \cap \mathcal{L}((\neg P)_A) = \emptyset$.
  - We will introduce the synchronized product automaton $A \otimes B$.
    - A transition of $A \otimes B$ represents a simultaneous transition of $A$ and $B$.
  - Property: $\mathcal{L}(A) \cap \mathcal{L}(B) = \mathcal{L}(A \otimes B)$.
- Final Problem: $\mathcal{L}(S_A \otimes (\neg P)_A) = \emptyset$.
  - We have to check whether the language of this automaton is empty.
  - We have to look for a word $w$ accepted by this automaton.
    - If no such $w$ exists, then $S \models P$.
    - If such a $w = w(r_l)$ exists, then $r$ is a counterexample, i.e. a run of $S$ such that $r \not\models P$.



## Synchronized Product of Two Automata

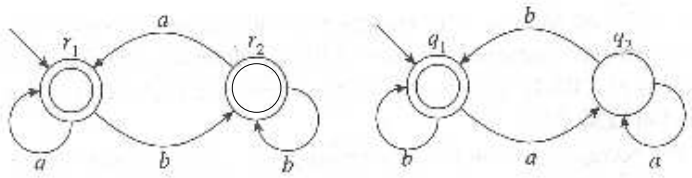


Given two finite automata $A = \langle I_A, R_A, State_A \rangle$ and $B = \langle I_B, R_B, F_B \rangle$.

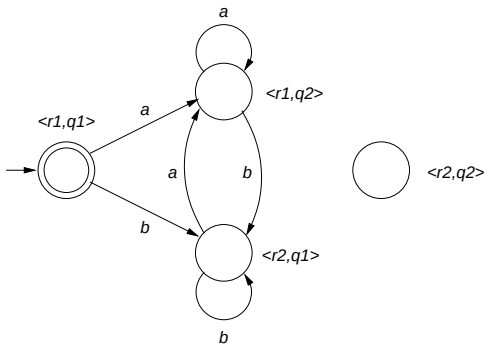
- Synchronized product $A \otimes B = \langle I, R, F \rangle$.
  - $State := State_A \times State_B$.
  - $Label := Label_A = Label_B$.
  - $I := I_A \times I_B$.
  - $R(I, \langle s_A, s_B \rangle, \langle s'_A, s'_B \rangle) :\Leftrightarrow R_A(I, s_A, s'_A) \wedge R_B(I, s_B, s'_B)$.
  - $F := State_A \times F_B$.

Special case where all states of automaton $A$ are accepting.

## Synchronized Product of Two Automata



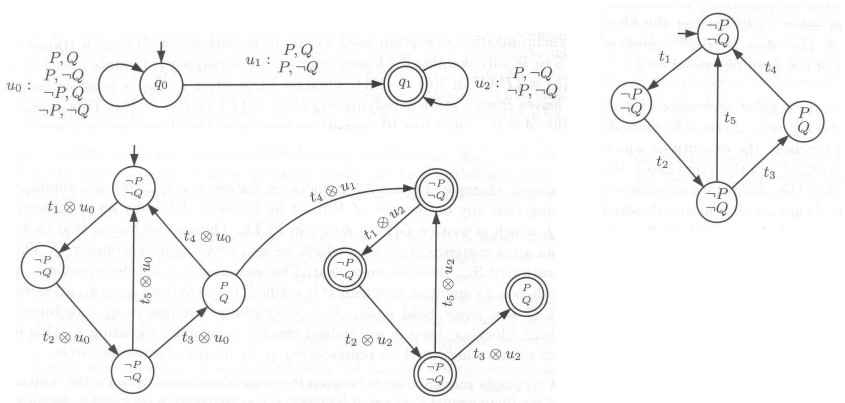

Edmund Clarke: "Model Checking", 1999.






## Example

Check whether $S \models \Box(P \Rightarrow \bigcirc\Diamond Q)$.




B. Berard et al: "Systems and Software Verification", 2001.

The product automaton accepts a run, thus the property does not hold.



## Checking Emptiness

How to check whether $\mathcal{L}(A)$ is non-empty?

- Suppose $A = \langle I, R, F \rangle$ accepts a run $r$.
  - Then $r$ contains infinitely many occurrences of some state in $F$.
  - Since *State* is finite, in some suffix $r'$ every state occurs infinit. often.
  - Thus every state in $r'$ is reachable from every other state in $r'$.
- $C$ is a strongly connected component (SCC) of graph $G$ if
  - $C$ is a subgraph of $G$,
  - every node in $C$ is reachable from every other node in $C$ along a path entirely contained in $C$, and
  - $C$ is maximal (not a subgraph of any other SCC of $G$).
- Thus the states in $r'$ are contained in an SCC $C$.
  - $C$ is reachable from an initial state.
  - $C$ contains an accepting state.
  - Conversely, *any* such SCC generates an accepting run.

$\mathcal{L}(A)$ is non-empty if and only if the reachability graph of $A$ has an SCC that contains an accepting state.



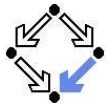
## Checking Emptiness

Find in the reachability graph an SCC that contains an accepting state.

- We have to find an accepting state with a cycle back to itself.
  - Any such state belongs to some SCC.
  - Any SCC with an accepting state has such a cycle.
  - Thus this is a sufficient and necessary condition.
- Any such a state $s$ defines a counterexample run $r$.
  - $r = \iota \rightarrow \ldots \rightarrow s \rightarrow \ldots \rightarrow s \rightarrow \ldots \rightarrow s \rightarrow \ldots$
  - Finite prefix $\iota \rightarrow \ldots \rightarrow s$ from initial state $\iota$ to $s$.
  - Infinite repetition of cycle $s \rightarrow \ldots \rightarrow s$ from $s$ to itself.

This is the core problem of PLTL model checking; it can be solved by a *depth-first search* algorithm.

## Basic Structure of Depth-First Search



Visit all states of the reachability graph of an automaton $\langle \{\iota\}, R, F \rangle$.

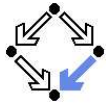
```
global
  StateSpace V := {}
  Stack D := ⟨⟩

proc main()
  push(D, ι)
  visit(ι)
  pop(D)
end
```

```
proc visit(s)
  V := V ∪ {s}
  for ⟨l, s, s'⟩ ∈ R do
    if s' ∉ V
      push(D, s')
      visit(s')
      pop(D)
    end
  end
end
```

State space $V$ holds all states visited so far; stack $D$ holds path from initial state to currently visited state.



## Checking State Properties



Apply depth-first search to checking a state property (assertion).

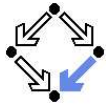
```
global
  StateSpace V := {}
  Stack D := ⟨⟩

proc main()
  // r becomes true, iff
  // counterexample run is found
  push(D, ι)
  r := search(ι)
  pop(D)
end
```

```
function search(s)
  V := V ∪ {s}
  if ¬check(s) then
    print D
    return true
  end
  for ⟨l, s, s'⟩ ∈ R do
    if s' ∉ V
      push(D, s')
      r := search(s')
      pop(D)
      if r then return true end
    end
  end
  return false
end
```

Stack $D$ can be used to print counterexample run.



## Depth-First Search for Acceptance Cycle

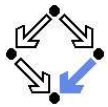


```
global
  ...
  Stack C := ⟨⟩

proc main()
  push(D, ι); r := search(ι); pop(D)
end

function searchCycle(s)
  for ⟨l, s, s'⟩ ∈ R do
    if has(D, s') then
      print D; print C; print s'
      return true
    else if ¬has(C, s') then
      push(C, s');
      r := searchCycle(s')
      pop(C);
      if r then return true end
    end
  end
  return false
end
```

```
boolean search(s)
  V := V ∪ {s}
  for ⟨l, s, s'⟩ ∈ R do
    if s' ∉ V
      push(D, s')
      r := search(s')
      pop(D)
      if r then return true end
    end
  end
  if s ∈ F then
    r := searchCycle(s)
    if r then return true end
  end
  return false
end
```



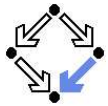
## Depth-First Search for Acceptance Cycle



- At each call of $search(s)$,
  - $s$ is a reachable state,
  - $D$ describes a path from $\iota$ to $s$.
- $search$ calls $searchCycle(s)$
  - on a reachable accepting state $s$
  - in order to find a cycle from $s$ to itself.
- At each call of $searchCycle(s)$,
  - $s$ is a state reachable from a reachable accepting state $s_a$,
  - $D$ describes a path from $\iota$ to $s_a$,
  - $D \rightarrow C$ describes a path from $\iota$ to $s$ (via $s_a$).
- Thus we have found an accepting cycle $D \rightarrow C \rightarrow s'$, if
  - there is a transition $s \xrightarrow{l} s'$,
  - such that $s'$ is contained in $D$.

If the algorithm returns "true", there exists a violating run; the converse follows from the exhaustiveness of the search.

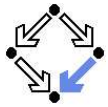
## Implementing the Search



- The state space $V$,
  - is implemented by a hash table for efficiently checking $s' \notin V$.
- Rather than using explicit stacks $D$ and $C$,
  - each state node has two bits $d$ and $c$,
  - $d$ is set to denote that the state is in stack $D$,
  - $c$ is set to denote that the state is in stack $C$.
- The counterexample is printed,
  - by searching, starting with $\iota$, the unique sequence of reachable nodes where $d$ is set until the accepting node $s_a$ is found, and
  - by searching, starting with a successor of $s_a$, the unique sequence of reachable nodes where $c$ is set until the cycle is detected.
- Furthermore, it is not necessary to reset the $c$ bits, because
  - *search* first explores all states reachable by an accepting state $s$ before trying to find a cycle from $s$; from this, one can show that
  - called with the first accepting node $s$ that is reachable from itself, *search2* will not encounter nodes with $c$ bits set in previous searches.
  - With this improvement, every state is only visited twice.



## Complexity of the Search



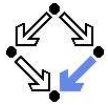
The complexity of checking $S \models P$ is as follows.

- Let $|P|$ denote the number of subformulas of $P$.
- $|State_{(\neg P)_A}| = O(2^{|P|})$.
- $|State_{A \otimes B}| = |State_A| \cdot |State_B|$.
- $|State_{S_A \otimes (\neg P)_A}| = O(|State_{S_A}| \cdot 2^{|P|})$
- The time complexity of *search* is linear in the size of *State*.
  - Actually, in the number of reachable states (typically much smaller).
  - Only true for the improved variant where the $c$ bits are not reset.
  - Then every state is visited at most twice.

PLTL model checking is linear in the number of reachable states but exponential in the size of the formula.



## Adding Weak Scheduling Fairness



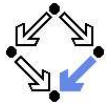
Assume that system is an asynchronous composition of $k$ components.

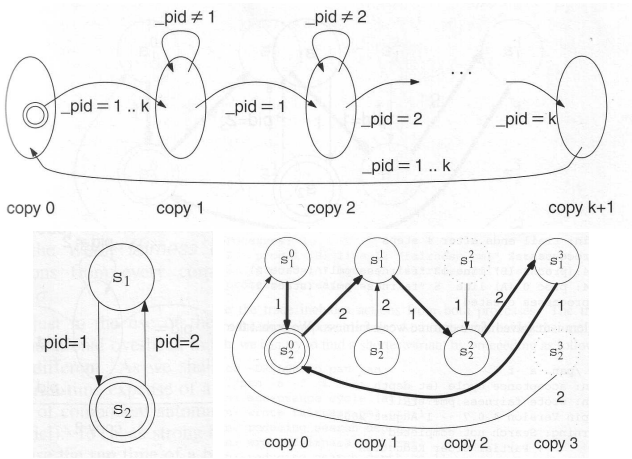
- Weak scheduling fairness:
  - A run satisfies w.s.f., if it has infinitely many transitions from every component which is permanently ready to perform some transition.
- Implementation in a model checker.
  - Construct $k+2$ copies $0, \ldots, k+1$ of the state space.
    - Actually just $k+2$ bits need to be added to every state.
    - Only copy 0 has the original acceptance states.
  - In copy 0, the destinations of all transitions from acceptance states are replaced by the corresponding states in copy 1.
  - In copy $i, 1 \leq i \leq k$, the destinations of all transitions of component $i$ are replaced by the corresponding states in copy $i+1$; a null transition is added from every state to the corresponding state in copy $i+1$ which is enabled if no other transition of component $i$ is enabled.
  - In copy $k+1$, the transitions of all components are replaced by the corresponding states in copy 0.

Every accepted run has (possibly null) transitions from all components.



## Example

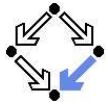





Gerard Holzmann: "The Spin Model Checker", 2004.

Runtime complexity of model checking is increased by a factor $k+2$.

## Summary

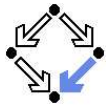


Basic PLTL model checking for deciding $S \models P$.

- Convert system $S$ to automaton $S_A$.
  - Atomic propositions of PLTL formula are evaluated on each state.
- Convert negation of PLTL formula $P$ to automaton $(\neg P)_A$.
  - How to do so, remains to be described.
- Construct synchronized product automaton $S_A \otimes (\neg P)_A$.
  - After that, formula labels are not needed any more.
- Find SCC in reachability-graph of product automaton.
  - A purely graph-theoretical problem that can be efficiently solved.
  - Time complexity is linear in the size of the state space of the system but exponential in the size of the formula to be checked.
  - Weak scheduling fairness with $k$ components: runtime is increased by factor $k + 2$ (worst-case, "in practice just factor 2" [Holzmann]).

The basic approach immediately leads to *state space explosion*; further improvements are needed to make it practical.

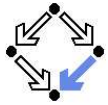








## From a PLTL Formula to an Automaton



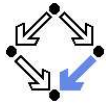
How to translate an PLTL formula $P$ to an automaton $P_A$?

- Automaton states: "atoms".
  - An atom is a particular set of subformulas of $P$.
    - May influence the value of the PLTL formula.
    - May be simultaneously satisfied by some state.
- Construction: $P_A$ is the synchronous product of two automata.
  - A local automaton: checking the safety aspect of $P$.
    - Atoms describe system states.
    - Transitions describe the system transitions allowed by $P$.
  - An eventuality automaton: checking the liveness aspect of $P$.
    - Atoms describe goals that have to be satisfied by a system run.
    - Transitions describe the changes of these goals.

Today there are better approaches generating smaller automata, but the presented one is easier to understand.



## The Closure of a Formula

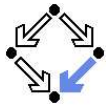


- We will only consider the operators $\neg, \wedge, \bigcirc, \mathbf{U}$.
  - $f_0 \vee f_1 \Leftrightarrow \neg(\neg f_0 \wedge \neg f_1)$, ....
  - $\Diamond f \Leftrightarrow \text{true}\ \mathbf{U}\ f$.
  - $\Box f \Leftrightarrow \neg\Diamond\neg f$.
- The closure $\mathcal{C}(P)$ of a PLTL formula $P$.
  - The smallest set satisfying the following conditions:
    - $P \in \mathcal{C}(P)$.
    - If $\neg f_0 \in \mathcal{C}(P)$, then $f_0 \in \mathcal{C}(P)$.
    - If $f_0 (\neq \neg f_1) \in \mathcal{C}(P)$, then $\neg f_0 \in \mathcal{C}(P)$.
    - If $f_0 \wedge f_1 \in \mathcal{C}(P)$, then $f_0, f_1 \in \mathcal{C}(P)$.
    - If $\bigcirc f_0 \in \mathcal{C}(P)$, then $f_0 \in \mathcal{C}(P)$.
    - If $f_0\ \mathbf{U}\ f_1 \in \mathcal{C}(P)$, then $f_0, f_1 \in \mathcal{C}(P)$.
- Example: $P :\Leftrightarrow (\neg p)\ \mathbf{U}\ q$.

  $\mathcal{C}(P) = \{P, \neg P, p, \neg p, q, \neg q\}$.

The closure of $P$ is the set of its subformulas and their negation.

## The Atoms of a Formula



The closure $\mathcal{C}(P)$ gives the material to construct sets of formulas.

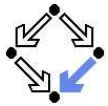
- An atom $K \subseteq \mathcal{C}(P)$ of $P$:
  - An atom $K$ of $P$ is a subset of the subformulas of $P$ that do not have a propositional contradiction.
    - If $f_0 \in \mathcal{C}(P)$, then $f_0 \in K$ iff $\neg f_0 \notin K$.
    - If $f_0 \wedge f_1 \in \mathcal{C}(P)$, then $f_0 \wedge f_1 \in K$ iff $f_0 \in K$ and $f_1 \in K$.
  - $P_K := \{K : K \text{ is an atom of } P\}$.
    - The set of atoms of $P$.
- Example: $P :\Leftrightarrow (\neg p)\ \mathbf{U}\ q$.

  $P_K = \{\{P, p, q\}, \{P, p, \neg q\}, \{P, \neg p, q\}, \{P, \neg p, \neg q\},$
  $\{\neg P, p, q\}, \{\neg P, p, \neg q\}, \{\neg P, \neg p, q\}, \{\neg P, \neg p, \neg q\}\}$.

The atoms represents the states of the automaton to be constructed.



## The Local Automaton of a Formula

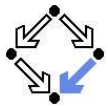


- The local automaton $P_L = \langle I_L, R_L, F_L \rangle$ of $P$:
  - $State := P_K$, $Label := \mathbb{P}(\mathcal{C}(P))$.
    - States are atoms, labels are sets of formulas from the closure of $P$.
  - $I_L := \{a \in P_K : P \in a\}$.
    - The initial states are all atoms that contain $P$.
  - $R_L(I, a, a') :\Leftrightarrow I = a \wedge$
    $\forall\ \bigcirc f \in \mathcal{C}(P) :$
    $\bigcirc f \in a \Leftrightarrow f \in a' \wedge$
    $\forall\ f_0 \mathbf{U} f_1 \in \mathcal{C}(P) :$
    $f_0 \mathbf{U} f_1 \in a \Leftrightarrow f_1 \in a \vee (f_0 \in a \wedge f_0 \mathbf{U} f_1 \in a')$.
    - $\bigcirc f$ holds now iff $f$ holds next.
    - $f_0\ \mathbf{U}\ f_1$ holds now iff $f_1$ holds now or $f_0\ \mathbf{U}\ f_1$ holds next.
  - $F_L := P_K$.
    - All states are accepting; i.e. every accepted run passes some state infinitely often.

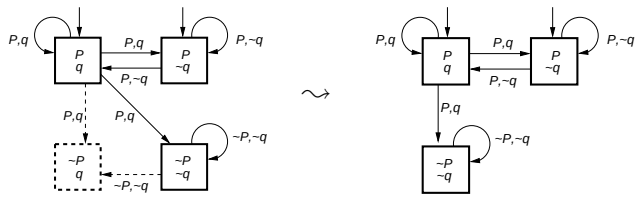
The transition label describes the obligations of the predecessor state; the local automaton checks all obligations except for the **U** formulas.



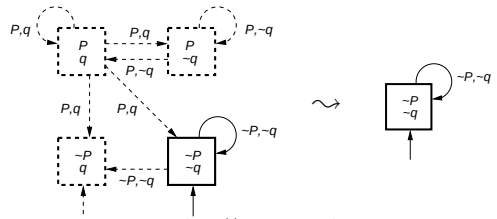
## Example



- Local automaton $P_L$ where $P \Leftrightarrow \text{true}\ \mathbf{U}\ q$ $(\Leftrightarrow \Diamond q)$.
  - Accepts every run (no safety aspect).

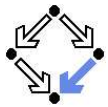



- Local automaton $(\neg P)_L$ where $\neg P \Leftrightarrow \neg(\text{true}\ \mathbf{U}\ q)$ $(\Leftrightarrow \Box \neg q)$.
  - Accepts only runs where $q$ does not occur.





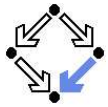

## The Eventuality Automaton of a Formula



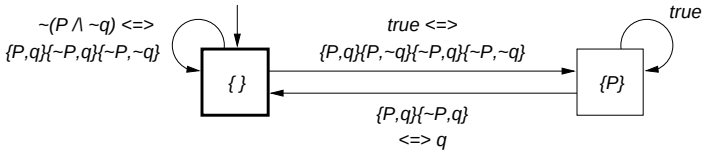
- The eventuality automaton $P_E = \langle I_E, R_E, F_E \rangle$ of $P$:
  - $State := \mathbb{P}(\{f_0 \mathbf{U} f_1 : f_0 \mathbf{U} f_1 \in P_K\})$; $Label := \mathbb{P}(\mathcal{C}(P))$.
    - States are sets of **U** formulas; labels are formula sets.
  - $I_L := \{\emptyset\}$.
    - The initial state is the empty formula set.
  - $R_L(I, a, a') :\Leftrightarrow$
    $(a = \emptyset \wedge \forall\ f_0 \mathbf{U} f_1 \in I : f_1 \in I \vee f_0 \mathbf{U} f_1 \in a')\ \vee$
    $(a \neq \emptyset \wedge \forall\ f_0 \mathbf{U} f_1 \in a : f_1 \in I \vee f_0 \mathbf{U} f_1 \in a')$.
    - From the initial state, the successor state has to fulfill all obligations of the label not satisfied by the label itself.
    - From each other state, the successor state has to fulfill all obligations of the current state not satisfied by the label.
  - $F_L := \{\emptyset\}$.
    - Only the initial state is accepting; i.e. every accepted run passes the initial state infinitely often.

The transition label describes the obligations of the predecessor state involving **U** formulas; the eventuality automaton checks these obligations.

## Example



- Eventuality automaton $P_E$ where $P \Leftrightarrow \text{true } \mathbf{U}\ q$ ($\Leftrightarrow \Diamond q$)
  (also for $(\neg P)_E$ where $\neg P \Leftrightarrow \neg(\text{true } \mathbf{U}\ q) \Leftrightarrow \Box\neg q$).

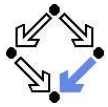



- If the initial obligation is $\{P, \neg q\}$, the only possible transition is to the non-accepting state.
  - In all other cases, there is an accepting run through the initial state.
- If the non-accepting state eventually encounters $q$, the obligation is fulfilled and a transition back to the accepting state is possible.

Only runs are accepted where all obligations are infinitely often fulfilled.



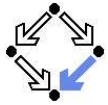
## Synchronous Product of Both Automata



- Synchronous product $P_L \otimes P_E$.
  - As already defined; finally all non-atomic propositions are removed from the labels.
- Intuitive understanding:
  - $P_L \otimes P_E$ accepts only runs that both $P_L$ and $P_E$ accept.
  - $P_L$ accepts a run with transition $a \xrightarrow{a} a'$, if the transition does not violate the safety aspect of $P$, even if $a$ does not satisfy the $\mathbf{U}$ formulas with which it was labeled.
  - $P_E$ accepts a run with transition $b \xrightarrow{a} b'$, if either $b$ satisfies the $\mathbf{U}$ formulas in label $a$, or if this obligation is passed on to $a'$ and eventually a state is reached that fulfills the obligation.
  - $P_L \otimes P_E$ thus accepts a run with transition $\langle a, b\rangle \xrightarrow{a} \langle a', b'\rangle$, if the transition does not violate the safety aspect of $P$ and if either $\langle a, b\rangle$ satisfies the $\mathbf{U}$ formulas in $a$, or if this obligation is passed on to $\langle a', b'\rangle$ and eventually a state is reached that fulfills the obligation.

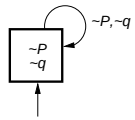


## Example

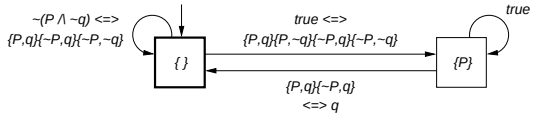


Product $(\neg P)_L \otimes (\neg P)_E$ where $\neg P \Leftrightarrow \neg(\text{true } \mathbf{U}\ q) \Leftrightarrow \Box\neg q$.

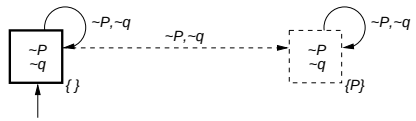
$(\neg P)_L$ :




$(\neg P)_E$ :



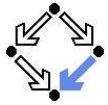

$(\neg P)_L \otimes (\neg P)_E$ :




$(\neg P)_L \otimes (\neg P)_E$ is equivalent to $(\neg P)_L$ (no liveness aspect in $\neg P$).

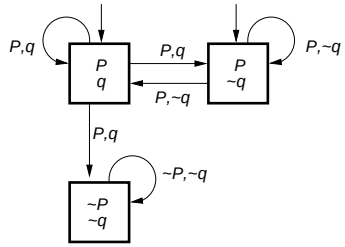


## Example

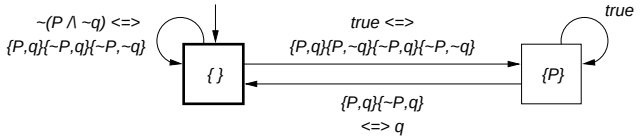


Product $P_L \otimes P_E$ where $P \Leftrightarrow \text{true } \mathbf{U}\ q \Leftrightarrow \Diamond q$.

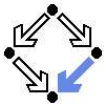
$P_L$ :



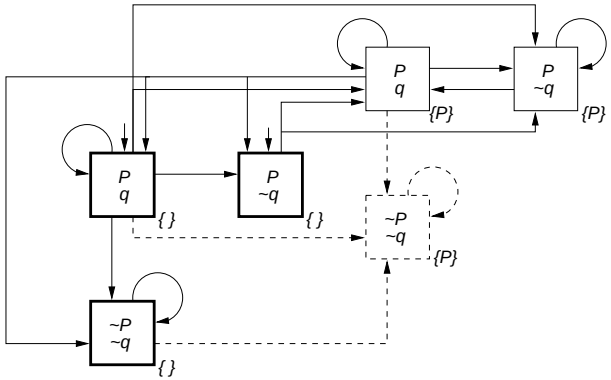

$P_E$ :



## Example



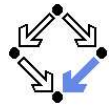
Product $P_L \otimes P_E$ where $P \Leftrightarrow \text{true } \mathbf{U}\ q \Leftrightarrow \Diamond q$.




Can remove a state from which no acceptance state can be reached.

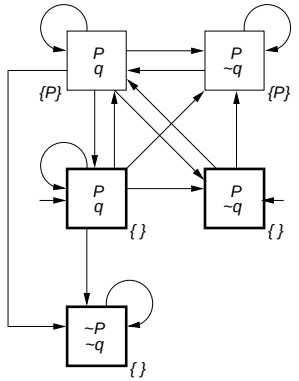


## Example



Product $P_L \otimes P_E$ where $P \Leftrightarrow \text{true } \mathbf{U}\ q \Leftrightarrow \Diamond q$.

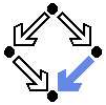
$P_L \otimes P_E$ :




Transitions are labeled with the atomic propositions of predecessor state.

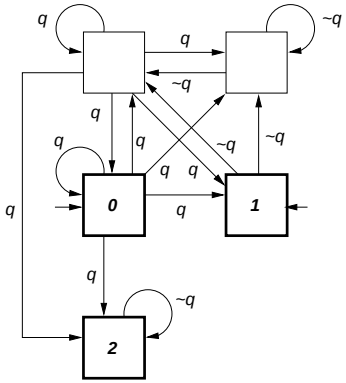


## Example



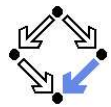
Product $P_L \otimes P_E$ where $P \Leftrightarrow \text{true } \mathbf{U}\ q \Leftrightarrow \Diamond q$.

- In every accepting run, $q$ must occur.
  - Accepting states have only incoming edges labeled with $q$.
  - Exception: $2 \rightarrow 2$, but 2 is not initial.
- All runs where $q$ occurs are accepted.
  - $q$ in first state: $0 \rightarrow \ldots$
  - $q$ not in first state: $1 \rightarrow \ldots$
  - Occurrences of $q$: $\ldots \rightarrow 0 \rightarrow \ldots \rightarrow 0 \rightarrow \ldots$
  - $q$ finitely often: $\ldots \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow \ldots$
  - $q$ eventually forever: $\ldots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \ldots$

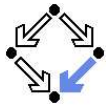



This automaton is much bigger than necessary (typical for the approach).





## On the Fly Model Checking



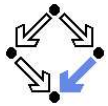
For checking $\mathcal{L}(S_A \otimes (\neg P)_A) = \emptyset$, it is not necessary to construct the states of $S_A$ in advance.

- Only the property automaton $(\neg P)_A$ is constructed in advance.
  - This automaton has comparatively small state space.
- The system automaton $S_A$ is constructed on the fly.
  - Construction is guided by $(\neg P)_A$ while computing $S_A \otimes (\neg P)_A$.
  - Only that part of the reachability graph of $S_A$ is expanded that is consistent with $(\neg P)_A$ (i.e. can lead to a counterexample run).
- Typically only a part of the state space of $S_A$ is investigated.
  - A smaller part, if a counterexample run is detected early.
  - A larger part, if no counterexample run is detected.

Unreachable system states and system states that are not along possible counterexample runs are never constructed.



## On the Fly Model Checking



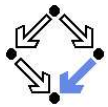
Expansion of state $s = \langle s_0, s_1 \rangle$ of product automaton $S_A \otimes (\neg P)_A$ into the set $R(s)$ of transitions from $s$ (**for** $\langle l, s, s' \rangle \in R(s)$ **do** ...).

- Let $S'_1$ be the set of all successors of state $s_1$ of $(\neg P)_A$.
  - Property automaton $(\neg P)_A$ has been precomputed.
- Let $S'_0$ be the set of all successors of state $s_0$ of $S_A$.
  - Computed on the fly by applying system transition relation to $s_0$.
- $R(s) := \{\langle l, \langle s_0, s_1 \rangle, \langle s'_0, s'_1 \rangle\rangle : s'_0 \in S'_0 \wedge s'_1 \in S'_1 \wedge s_1 \xrightarrow{l} s'_1 \wedge L(s'_0) \in l\}$.
  - Choose candidate $s'_0 \in S'_0$.
  - Determine set of atomic propositions $L(s'_0)$ true in $s'_0$.
  - If $L(s'_0)$ is not consistent with the label of any transition $\langle s_0, s_1 \rangle \xrightarrow{l} \langle s'_0, s'_1 \rangle$ of the proposition automaton, $s'_0$ it is ignored.
  - Otherwise, $R$ is extended by every transition $\langle s_0, s_1 \rangle \xrightarrow{l} \langle s'_0, s'_1 \rangle$ where $L(s'_0)$ is consistent with label $l$ of transition $s_1 \xrightarrow{l} s'_1$.

Actually, depth-first search proceeds with first suitable successor $\langle s'_0, s'_1 \rangle$ before expanding the other candidates.



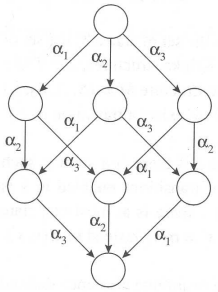
## Partial Order Reduction



Core problem of model checking: state space explosion.

- Take asynchronous composition $S_0||S_1||\ldots||S_{k-1}$.
  - Take state $s$ where one transition of each component is enabled.
    - Assume that the transition of one component does not disable the transitions of the other components and that no other transition becomes enabled before all three transitions have been performed.
  - Take state $s'$ after execution of all three transitions.
    - There are $k!$ paths leading from $s$ to $s'$.
    - There are $2^k$ states involved in the transitions.

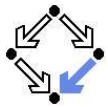
Sometimes it suffices to consider a *single path* with $k + 1$ states.



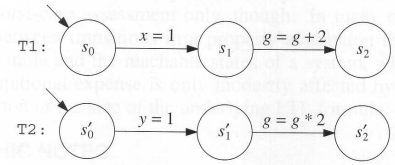

Edmund Clarke: "Model Checking", 1999.

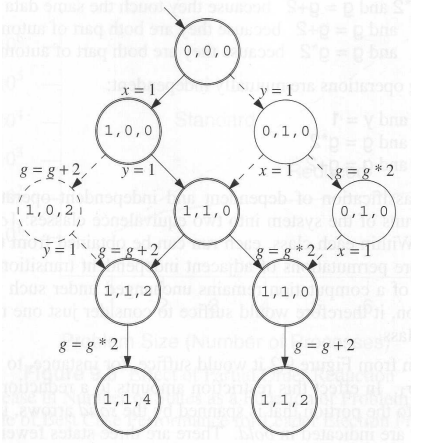


## Example



Check $(T1||T2) \models \Diamond g \geq 2$.

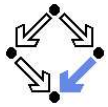



Gerard Holzmann: "The Spin Model Checker", 1999.

For checking $\Diamond g \geq 2$, it suffices to check only one ordering of the independent transitions $x = 1$ and $y = 1$ (not true for checking $\Box x \geq y$).

## Partial Order Reduction



Check $S \models P$.

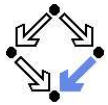
**boolean** *search*$(s)$ ... **for** $\langle l, s, s' \rangle \in R(s)$ **do** $\rightsquigarrow$ **boolean** *search*$(s)$ ... **for** $\langle l, s, s' \rangle \in ample_P(s)$ **do**

- $ample_P(s) \subseteq R(s)$.
  - The ample set $ample_P(s)$.
    - The set of transitions from $s$ to be considered for checking $P$.
  - $R(s) := \{\langle l, s, s' \rangle : l \in Label \wedge s' \in State\}$.
    - The set of all transitions from $s$.
  - Optimization: $ample_P(s) \subsetneq R(s)$.
    - Search space is reduced.

We will now investigate the calculation of the ample set.

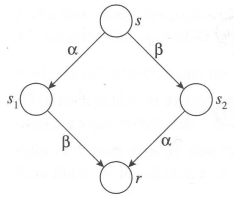


## Independence



When can two transitions of two components be executed in any order?

- Independence relation $I \subseteq Trans \times Trans$.
  - $I$ is a relation between transitions.
    - $Trans := Label \times State \times State$.
    - $I(\alpha, \beta)$ ... $\alpha$ and $\beta$ are independent.
  - $I$ satisfies the following conditions for every $s \in State$ and $\alpha, \beta \in Trans$:
    - $I$ is antireflexive and symmetric: $\neg I(\alpha, \alpha) \wedge (I(\alpha, \beta) \Rightarrow I(\beta, \alpha))$.
    - Enabledness: $\alpha, \beta \in R(s) \Rightarrow \alpha \in R(\beta(s))$. If $\alpha$ and $\beta$ are enabled, then $\alpha$ is still enabled after executing $\beta$.
    - Commutativity: $\alpha, \beta \in R(s) \Rightarrow \alpha(\beta(s)) = \beta(\alpha(s))$. If $\alpha$ and $\beta$ are enabled, then they may be executed in any order.
- Dependence relation $D(\alpha, \beta) :\Leftrightarrow \neg I(\alpha, \beta)$.
  - Two transitions are dependent, if they are not independent.



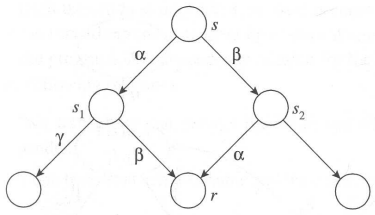

Edmund Clarke: "Model Checking", 1999.

$I(\alpha, \beta)$ is necessary for considering only one transition $\alpha$ or $\beta$ from $s$.



## Open Problems






Edmund Clarke: "Model Checking", 1999.

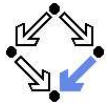
$I(\alpha, \beta)$ is not sufficient for considering only one transition $\alpha$ or $\beta$.

- Problem 1: property $P$ may distinguish between $s_1$ and $s_2$.
  - $s \rightarrow s_1 \rightarrow r$ may be recognized as different from $s \rightarrow s_2 \rightarrow r$.
- Problem 2: states $s_1$ and $s_2$ may have successors in addition to $r$.
  - If $\alpha$ is not considered, not every successor of $s_1$ may be explored.

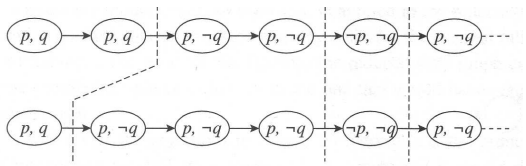
Need some more checks to make sure that these problems do not occur.



## Invisibility and Stuttering Equivalence

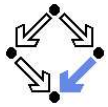


- A transition $\alpha$ is invisible:
  - $\forall s, s' \in State : s' = \alpha(s) \Rightarrow L(s) = L(s')$.
  - Predecessor and successor state fulfill the same atomic propositions.
- Two runs $r$ and $s$ are stuttering equivalent:
  - There are two infinite sequences $i_0, i_1, \ldots$ and $j_0, j_1$ of positions in $r$ respectively $s$ such that for all $k \in \mathbb{N}$
    $L(r(i_k)) = L(r(i_k + 1)) = \ldots = L(r(i_{k+1} - 1)) =$
    $L(s(j_k)) = L(s(j_k + 1)) = \ldots = L(s(j_{k+1} - 1))$.
  - $r$ and $s$ have identical sequences of labels except for the number of occurrences of each label in subsequent positions.




Edmund Clarke: "Model Checking", 1999.

## Stuttering Invariance

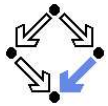


- A PLTL formula $P$ is invariant under stuttering:
  - For each pair of stuttering equivalent runs $r$ and $s$: $r \models P$ iff $s \models P$.
  - $P$ cannot distinguish among $r$ and $s$.
- Every PLTL formula without $\bigcirc$ is invariant under stuttering.
  - Only temporal operators $\Box, \Diamond, \mathbf{U}$.
  - Only these operators are allowed in Spin as well.
  - Also converse of statement is true: every path property that does not distinguish between stuttering equivalent runs can be expressed by a PLTL formula without $\bigcirc$.

Partial order reduction only applies to checking formulas that are invariant under stuttering.



## Computation of the Ample Set



Check $(S_0 || \ldots || S_{k-1}) \models P$ where $P$ is invariant under stuttering.

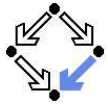
```
function ample_P(s) :
  for S_i with R_i(s) ≠ ∅ do
    if C1(R_i(s)) ∧ C2(s, R_i(s)) ∧ C3(s, S_i) then
      return R_i(s)
    end
  end
  return R(s)
end
```

- Transition relation $R_i$ of component $i$:
  - Total system transition relation $R := \bigcup_{0 \le i < k} R_i$.
- Result of $ample_P(s)$:
  - Some component $i$'s non-empty set of transitions from $s$ that satisfies conditions $C1$, $C2$, $C3$ (if such a set exists).
  - All transitions from $s$, if no such set exists.

If successful, only transitions from component $i$ are considered.



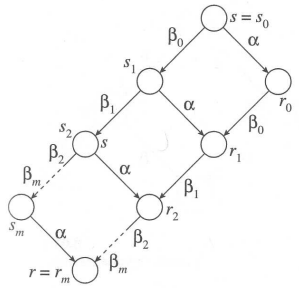
## Condition 1: Invisibility



$C1(T) :\Leftrightarrow$
$\forall \alpha \in T : \alpha$ *is invisible*.

- If $ample(s) \neq R(s)$, then every $\alpha \in ample(s)$ is invisible.

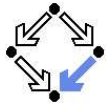
A reduced ample set has only invisible transitions; otherwise the discarded runs would not be stuttering equivalent to the considered run.




Edmund Clarke: "Model Checking", 1999.

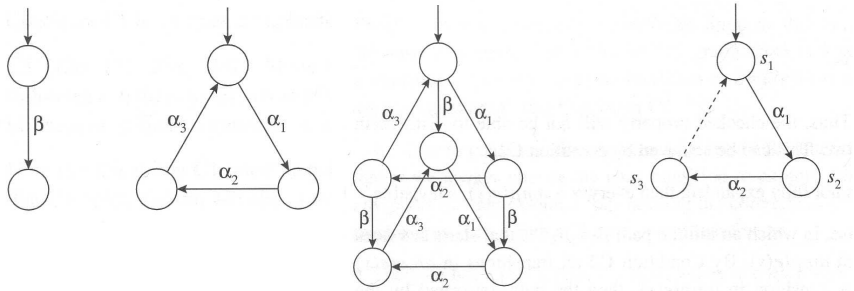


## Condition 2: Cycle Condition



$C2(s, T) :\Leftrightarrow \forall \alpha \in T : \neg has(D, \alpha(s))$.

- If $ample(s) \neq R(s)$, then no $\alpha \in ample(s)$ may lead to a cycle.




Edmund Clarke: "Model Checking", 1999.

C2 ensures that $ample_P(s_3) = \{\beta\}$; otherwise $\beta$ would not appear at all in the reduced state graph.

## Condition 3: Independence

$C3(s, S_i) :\Leftrightarrow$
$\forall P_j \neq P_i :$
$\forall \beta \in R_j(s) :$
$(\forall \alpha \in R_i(s) : I(\beta, \alpha) \wedge$
$\forall \alpha \in current_i(s) \backslash R_i(s) : \neg enable(\beta, \alpha))$

- $current_i(s) := \{\alpha : \alpha \in R_i \wedge \exists s' : \alpha \in R(s') \wedge pc_i(s) = pc_i(s')\}$.
  - The set of transitions of $S_i$ that are enabled in any state where $S_i$ is at the same statement as in $s$ (including those not enabled in $s$ itself).
- $enable(\beta, \alpha) :\Leftrightarrow \exists s : \alpha \notin R(s) \wedge \beta \in R(s) \wedge \alpha \in R(\beta(s))$
  - Transition $\beta$ may enable transition $\alpha$.

Every transition $\beta$ that may be executed before any transition $\alpha \in ample(s)$ is independent of $\alpha$ and cannot enable $\alpha$; thus it it safe to execute $\alpha$ first and $\beta$ later.



## Interpretation

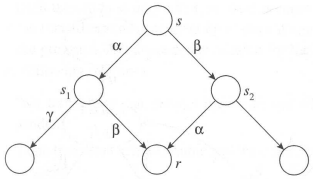
Which transitions depend on each other respectively enable each other?

- $D(\beta, \alpha)$ iff
  - $\beta$ and $\alpha$ belong to the same process or
  - $\beta$ and $\alpha$ share a variable that is changed by $\beta$ or $\alpha$, or
  - $\beta$ and $\alpha$ are two send statements or two receive statements on the same channel.
- $enable(\beta, \alpha)$ iff
  - $\beta$ and $\alpha$ belong to the same process and $\alpha$ is a possible successor statement of $\beta$, or
  - $\alpha$ is a statement that depends on the value of a shared variable and $\beta$ is a statement that may change this variable, or
  - $\beta$ and $\alpha$ are send respectively receive statements on the same channel.

For checking Condition 3, only certain pairs of transitions are of interest.



## Closing the Open Problems




Edmund Clarke: "Model Checking", 1999.

- Problem 1: property $P$ may distinguish between $s_1$ and $s_2$.
  - Not possible because of Condition 1 (Invisibility) which makes $s \rightarrow s_1 \rightarrow r$ and $s \rightarrow s_2 \rightarrow r$ stuttering equivalent.
- Problem 2: states $s_1$ and $s_2$ may have successors in addition to $r$.
  - Transition $\gamma$ enabled in $s_1$ is independent of $\beta$ (Condition 3) and, since enabled in $s_1$, thus also enabled in $r$.
  - Since $\beta$ is invisible (Condition 1), sequences $s \xrightarrow{\alpha} s_1 \xrightarrow{\gamma} s_1'$ and $s \xrightarrow{\beta} s_2 \xrightarrow{\alpha} r \xrightarrow{\gamma} r'$ are stuttering equivalent.



## Example

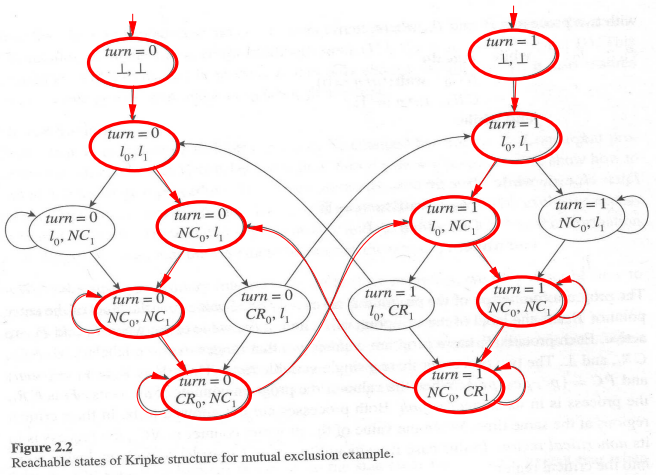
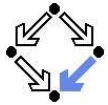



**Figure 2.2**
Reachable states of Kripke structure for mutual exclusion example.

Edmund Clarke et al: "Model Checking", 1999.

System after partial order reduction.

# Other Optimizations



- Statement merging.
  - Special case of partial order reduction where a sequence of transitions of same component is combined to a single transition.
- State compression.
  - Collapse compression: each state holds pointers to component states; thus component states can be shared among many system states.
  - Minimized automaton representation: represent state set $V$ not by hash table but by finite state automaton that accepts a state (sequence of bits) $s$ if and only if $s \in V$.
  - Hash compact: store in the hash table a hash value of the state (computed by a different hash function). Probabilistic approach: fails if two states are mapped to the same hash value.
  - Bitstate hashing: represent $V$ by a bit table whose size is much larger than the expected number of states; each state is then only represented by a single bit. Probabilistic approach: fails if two states are hashed to the same position in the table.