# More on Relations 2

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# Overview

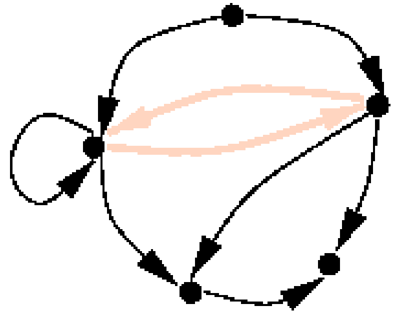
- Partial Order
- Quasi Order
- Total Order
- Hasse Diagrams
- Related Notions

# Antisymmetry

Definition: A binary relation is antisymmetric, if different elements are not mutually related.

$$
\begin{aligned}
&R \text{ is antisymmetric on } S :\Leftrightarrow \\
&\quad \forall x \in S, y \in S : (\langle x, y \rangle \in R \wedge \langle y, x \rangle \in R) \Rightarrow x = y.
\end{aligned}
$$

Example: the following relation is not antisymmetric:



# Partial Order

Definition: A binary relation on $S$ is a partial order, if it is reflexive, antisymmetric, and transitive:

$$\begin{aligned}
&R \text{ is partial order on } S :\Leftrightarrow \\
&\quad R \subseteq S \times S \land \\
&\quad R \text{ is reflexive on } S \land \\
&\quad R \text{ is antisymmetric on } S \land \\
&\quad R \text{ is transitive on } S.
\end{aligned}$$

Example:

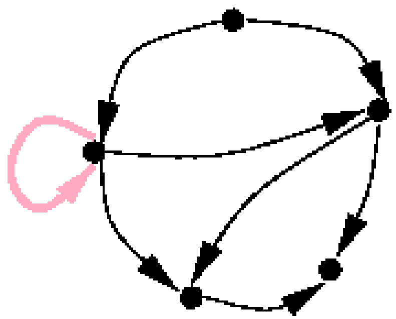
- $\subseteq$ is a partial order on $\mathbb{P}(S)$, for every set $S$.
- $|$ (divides) is a partial order on $\mathbb{N}$.

# Irreflexivity

Definition: A binary relation is irreflexive, if no element is related to itself:

$$\begin{gathered} R \text{ is irreflexive on } S :\Leftrightarrow \\ \forall x \in S : \langle x, x \rangle \notin R. \end{gathered}$$

Example: the following relation is not irreflexive:

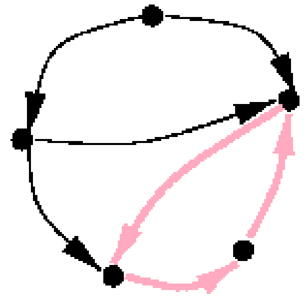


## Quasi Order

Definition: A binary relation on $S$ is a quasi order, if it is irreflexive and transitive:

$$
\begin{aligned}
&R \text{ is quasi order on } S :\Leftrightarrow \\
&\quad R \subseteq S \times S \land \\
&\quad R \text{ is irreflexive on } S \land \\
&\quad R \text{ is transitive on } S.
\end{aligned}
$$

Example: the following relation is not a quasi-order:



# Quasi Order from Partial Order

Proposition: Let $\preceq$ be a partial order. Then the relation $\prec$ defined as

$$x \prec y :\Leftrightarrow x \preceq y \wedge x \neq y.$$

is a quasi-order:

$$\begin{array}{l} \forall S, \preceq : \preceq \text{ is partial order on } S \Rightarrow \\ \quad \prec \text{ is quasi order on } S \end{array}$$

Proof: exercise.

## Quasi Orders and Anti-Symmetry

Proposition: Every quasi order is antisymmetric:

$$\forall S, \prec : \prec \text{ is quasi order on } S \Rightarrow$$
$$\prec \text{ is antisymmetric on } S.$$

Proof: Take arbitrary $S$ and quasi order $\prec$ on $S$. Assume there exist $x \in S$ and $y \in S$ with $x \neq y$ such that $x \prec y$ and $y \prec x$. By transitivity, we have $x \prec x$ which contradicts the irreflexivity of $\prec$.

Only difference between partial orders and quasi orders is reflexivity versus irreflexivity.

## Example

- $\leq$ is a partial order on $\mathbb{R}$.
- $<$ is a quasi-order on $\mathbb{R}$.
- $\subset$ (proper subset) is a quasi-order on $\mathbb{P}(S)$, for every set $S$.

## Comparability

Definition: Two elements $x$ and $y$ are incomparable with respect to a relation $\preceq$ if neither $x \preceq y$ nor $y \preceq x$:

$$
\begin{aligned}
&x \text{ and } y \text{ are incomparable w.r.t. } \preceq \;:\Leftrightarrow \\
&\quad x \not\preceq y \wedge y \not\preceq x.
\end{aligned}
$$

Example: $\{0,1\}$ and $\{1,2\}$ are incomparable w.r.t. $\subseteq$.

## Total Order

Definition: A partial order is a total order or linear order or chain if no elements are incomparable with respect to this order:

$$
\begin{aligned}
&R \text{ is total order on } S \;:\Leftrightarrow \\
&\quad R \text{ is partial order on } S \;\wedge \\
&\quad \neg\exists x \in S, y \in S : x \text{ and } y \text{ are incomparable w.r.t. } R.
\end{aligned}
$$

Example:

- $\subseteq$ is not a total order on $\mathbb{P}(\{0,1,2\})$.
- $\leq$ is a total order on $\mathbb{R}$.

## Example

The lexicographic order on $\mathbb{N} \times \mathbb{N}$ is a total order:

$$x \preceq y :\Leftrightarrow x_0 < y_0 \vee (x_0 = y_0 \wedge x_1 \leq y_1)$$

$$\langle 0,0\rangle \prec \langle 0,1\rangle \prec \langle 0,2\rangle \prec \ldots \prec \langle 1,0\rangle \prec \langle 1,1\rangle \prec \langle 1,2\rangle \prec \ldots$$

Comparison first by first component, then by second component.

# Example

Let $\preceq$ be a total order on an alphabet $A$, let $W_n := \mathbb{N}_n \rightarrow A$ be the set of all words with $n$ letters, and let

$$\overline{w} := \textbf{such } u : \exists n \in \mathbb{N} : w \in W_n \land u \in W_{n-1} \land \forall i \in \mathbb{N}_{n-1} : u_i = w_{i+1}$$

The lexicographic order $\preceq_n \subseteq W_n \times W_n$ defined as

$$w \preceq_n u :\Leftrightarrow n = 0 \lor w_0 \prec u_0 \lor (w_0 = u_0 \land \overline{w} \preceq_{n-1} \overline{u})$$

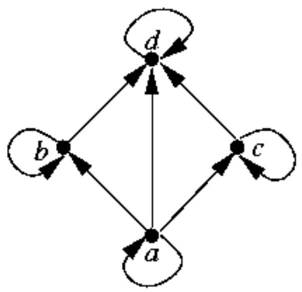
is a total order.

$$\text{“back”} \prec_4 \text{“bare”} \prec_4 \text{“base”} \prec_4 \text{“bear”} \prec_4 \text{“bend”} \prec_4 \text{“care”}$$

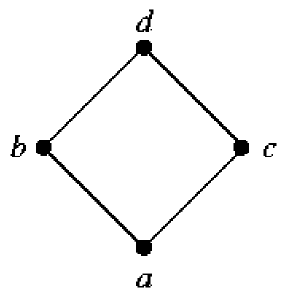
Compare letters in the order of their positions.

# Hasse Diagrams

## Directed Graph:

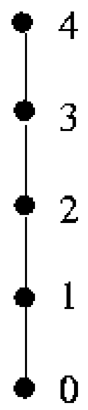



## Hasse Diagram:



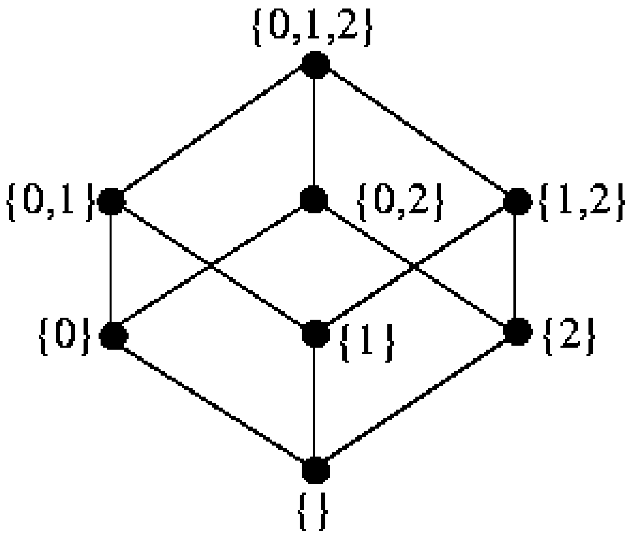
# Example

This Hasse diagram illustrates the total order $\leq$ on $\mathbb{N}_5$.



# Example

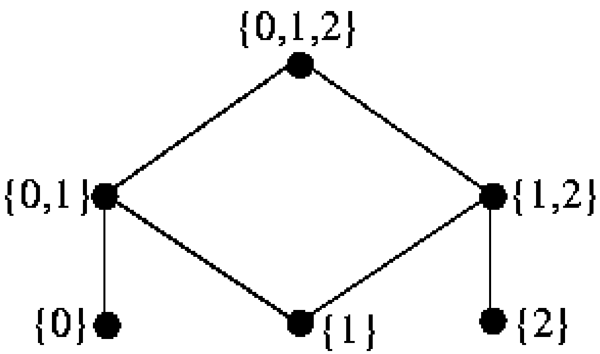
This Hasse diagram illustrates the partial order $\subseteq$ on $\mathbb{P}(\{1, 2, 3\})$:



# Example

This Hasse diagram illustrates the partial order $\subseteq$ on

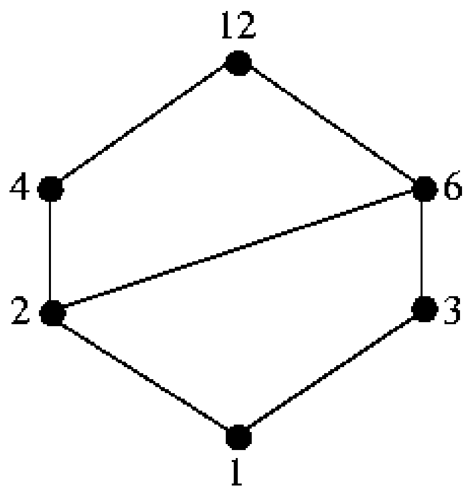
$$\{\{0\},\{1\},\{2\},\{0,1\},\{1,2\},\{0,1,2\}\}:$$



# Example

Let $F_n := \{m \in \mathbb{N} : m|n\}$ (the factors of $n$).

The following Hasse diagram illustrates the partial order $|$ on $F_{12}$:



## Least and Greatest Element

Definition: If $x$ is an element of $S$ such that $x$ is less than or equal to any element of $S$, then $x$ is called the least element of $S$:

$$\begin{aligned}&x \text{ is least element of } S \text{ w.r.t. } \preceq \; :\Leftrightarrow \\ &\quad x \in S \wedge \forall y \in S : x \preceq y.\end{aligned}$$

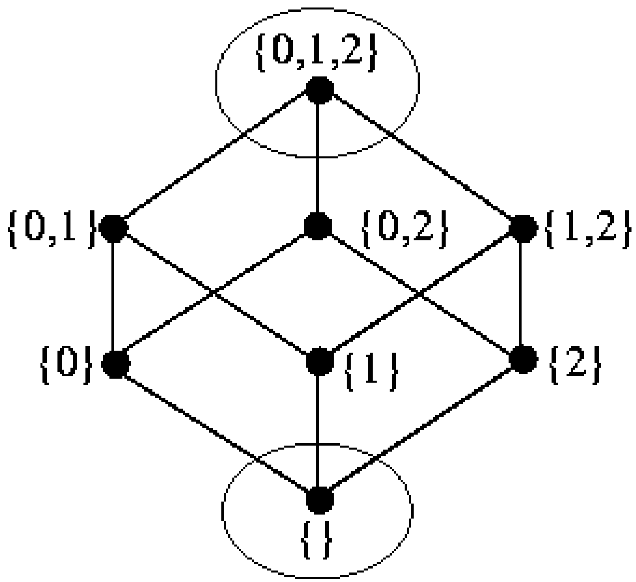
If $x$ is an element of $S$ such that $x$ is greater than or equal to any element of $S$, then $x$ is called the greatest element of $S$:

$$\begin{aligned}&x \text{ is greatest element of } S \text{ w.r.t. } \preceq \; :\Leftrightarrow \\ &\quad x \in S \wedge \forall y \in S : y \preceq x.\end{aligned}$$

Least respectively greatest element is unique (if it exists).

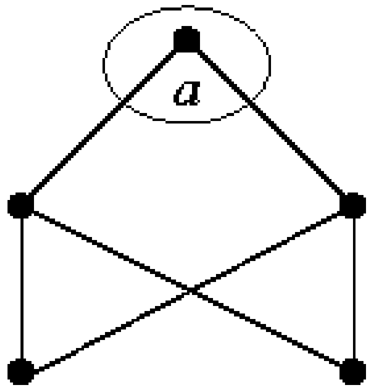
# Example

$\emptyset$ is the least element of $\mathbb{P}(\{1,2,3\})$ with respect to $\subseteq$, $\{1,2,3\}$ is the greatest element:



# Example

The following Hasse diagram denotes a partial order with greatest element $a$ but without a least element:



## Minimal and Maximal Element

Definition: If $x$ is an element of $S$ such that no element of $S$ is smaller than $x$, then $x$ is called a minimal element of $S$.

$$\begin{gathered}x \text{ is minimal element of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\ x \in S \wedge \forall y \in S : y \preceq x \Rightarrow y = x.\end{gathered}$$

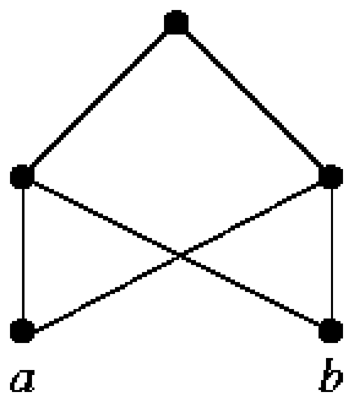
If $x$ is an element of $S$ such that no element of $S$ is greater than $x$, then $x$ is called a maximal element of $S$.

$$\begin{gathered}x \text{ is maximal element of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\ x \in S \wedge \forall y \in S : x \preceq y \Rightarrow x = y.\end{gathered}$$

Minimal and maximal elements are not necessarily unique.

## Example

The following Hasse diagram denotes a partial order with two minimal elements $a$ and $b$:



## Lower and Upper Bound

Definition: A value $x$ is a lower bound of $S$, if it is less than or equal to any element of $S$:

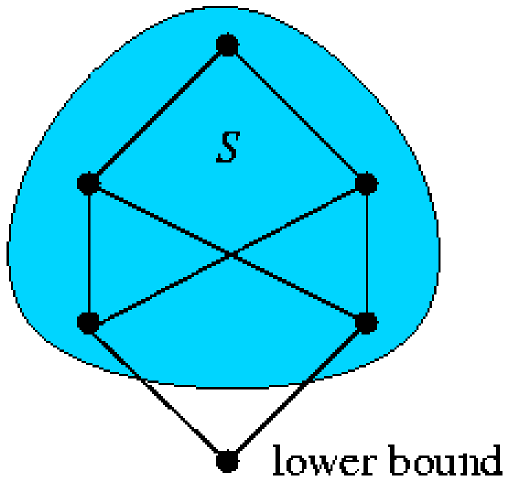
$$\begin{aligned}&x \text{ is lower bound of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\ &\quad \forall y \in S : x \preceq y.\end{aligned}$$

A value $x$ is an upper bound of $S$, if it is greater than or equal to any element of $S$:

$$\begin{aligned}&x \text{ is upper bound of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\ &\quad \forall y \in S : y \preceq x.\end{aligned}$$

A lower (upper) bound of a set $S$ need not be element of $S$.

# Example



## Infimum and Supremum

A value $x$ is a infimum of $S$, if it is the greatest lower bound of $S$:

$$
\begin{aligned}
&x \text{ is infimum of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\
&\quad x \text{ is lower bound of } S \text{ w.r.t. } \preceq \;\wedge\\
&\quad \forall y : y \text{ is lower bound of } S \text{ w.r.t. } \preceq \Rightarrow y \preceq x.
\end{aligned}
$$

A value $x$ is a supremum of $S$, if it is the least upper bound of $S$:

$$
\begin{aligned}
&x \text{ is supremum of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\
&\quad x \text{ is upper bound of } S \text{ w.r.t. } \preceq \;\wedge\\
&\quad \forall y : y \text{ is upper bound of } S \text{ w.r.t. } \preceq \Rightarrow x \preceq y.
\end{aligned}
$$

Infimum respectively supremum of a set $S$ need not be element of $S$.

## Order Laws

Proposition: Let $\preceq$ be a partial order on $S$. Then for every $x$, the following holds:

$$\begin{array}{l}
x \text{ is least (greatest) element of } S \text{ w.r.t. } \preceq \;\Rightarrow \\
\quad x \text{ is minimal (maximal) element of } S \text{ w.r.t. } \preceq, \\
x \text{ is least (greatest) element of } S \text{ w.r.t. } \preceq \;\Rightarrow \\
\quad x \text{ is infimum (supremum) of } S \text{ w.r.t. } \preceq, \\
x \text{ is lower (upper) bound of } S \text{ w.r.t. } \preceq \wedge\, x \in S \;\Rightarrow \\
\quad x \text{ is least (greatest) element of } S \text{ w.r.t. } \preceq .
\end{array}$$

## Summary

- Partial Order

  reflexive, antisymmetric, transitive.

- Quasi Order

  irreflexive, antisymmetric, transitive.

- Total Order

  Partial order that compares all elements.

- Hasse Diagram

  Compact visualization of partial order.

- Related Notions