The Language of Logic

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Overview

- Motivation and preliminaries.
- Propositional logic.
- Predicate logic.
- Example.

Motivation and Preliminaries

Motivation

- Precise language.
  - Formulating statements.
  - Designating objects.

  Resolve ambiguities.
- Intellectual framework.
  - Sound reasoning.
  - Sound arguing.

  Guide thinking.

Let's focus on the language aspect first.

Formality vs Informality

- Symbolic Arithmetic:

  "The product of x and of the sum of y and z if the sum of the products of x and y and of x and z":

$$x * (y + z) = x * y + x * z.$$

- Symbolic Logic:

  "Between every natural number and its double (the bounds included) there is at least one prime number":

$$\forall x : x \in \mathbb{N} \Rightarrow \exists y : x \leq y \land y \leq 2x \land y \text{ is prime.}$$

Formal notation is a tool to write clear and concise statements.

Preliminaries

- Truth values (Boolean values):

  true, false.

- Formula:

  Syntactic phrase whose semantics is a truth value

  - Syntax: external representation.
  - Semantics: underlying meaning.

$$\boxed{\text{Formula} \xrightarrow{\text{Meaning}} \text{Truth Value}}$$

Operational Interpretation

```
public interface Formula
{
  boolean eval() throws EvalException;
}

Formula formula = ...;
boolean meaning = formula.eval();
```

Evaluation of a formula returns a truth value (normally).

Equivalence of Formulas

$$A \text{ iff } B$$

- Formulas A and B have the same meaning in any context.
- Operationally, then the following always holds:

$$A\texttt{.eval() == }B\texttt{.eval()}$$

Propositional Logic

Propositional Logic

- Logic of formula composition:
  - Basic formulas are "black boxes": only truth value known.
  - Combination of simpler formulas to more complex ones.
- Logical Connective
  - Syntactic operator that combines formulas to a new formula.
  - F, T, $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

Rules for construction of formulas and computation of their meanings.

Syntax of Propositional Logic

Let A and B be formulas. Then the following are formulas:

- Logical constants: F, T.

  "false", "true".

- Negation: $(\neg A)$.

  "not A".

Connectives of arity 0 and 1.

Syntax of Propositional Logic (Continued)

- Conjunction: $(A \wedge B)$.

  "A and B".

- Disjunction: $(A \vee B)$.

  "A or B".

- Implication: $(A \Rightarrow B)$.

  "A implies B".

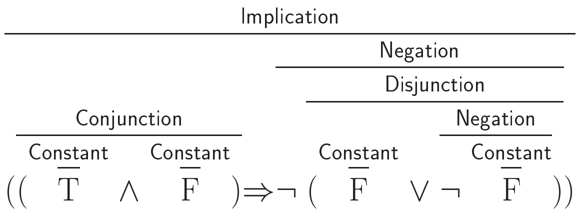
- Equivalence: $(A \Leftrightarrow B)$.

  "A is equivalent to B".

Connectives of arity 2.

Hierarchical Construction of Formulas

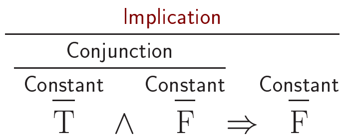
$$((\mathrm{T} \land \mathrm{F}) \Rightarrow \neg(\mathrm{F} \lor \neg \mathrm{F}))$$

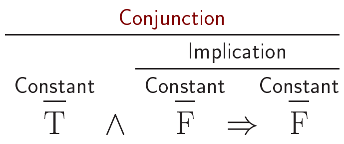


If syntactic structure clear, parentheses may be omitted.

Ambiguous Syntax

$$\mathrm{T} \land \mathrm{F} \Rightarrow \mathrm{F}$$





Logical Constants

$$\mathrm{F}, \mathrm{T}.$$

- Other syntactic forms:
  - 0, 1.
  - wrong, right.
  - incorrect, correct.
  - false, true.
- Semantics:
  - Meaning of F is false.
  - Meaning of T is true.

Operational Interpretation

```
public final class False implements Formula
{
  public boolean eval() throws EvalException
  {
    return false;
  }
}

public final class True implements Formula
{
  public boolean eval() throws EvalException
  {
    return true;
  }
}
```

Negations

$$(\neg A).$$

- Other syntactic forms:
  - $\overline{A}$, $\sim A$, $-A$, $!A$;
  - non-A, "not A", "never A", "in no case A";
  - `not(A)`.
- Semantics: $\neg A$ is true, if and only if A is false.

A	$\neg A$
false	true
true	false

Operational Interpretation

```
public final class Not implements Formula
{
  private Formula formula;

  ...

  public boolean eval() throws EvalException
  {
    if (formula.eval())
      return false;
    else
      return true;
  }
}
```


Inversion of Negation

Proposition: For every formula A, we have

$$\neg\neg A \text{ iff } A.$$

Proof: Let A be an arbitrary formula. We have to show that the meaning of $\neg\neg A$ is the same as the meaning of A, i.e., that they have the same truth values.

A	$\neg A$	$\neg\neg A$
false	true	false
true	false	true

Since the last column coincides with the first column, we are done.

Conjunctions

$$(A \wedge B)$$

- Other syntactic forms:
  - A, B; $A * B$; $A \& B$; A `&&` B;
  - "A and B", "A as well as B" ("sowohl A als auch B");
  - `and(A, B)`.
- Semantics: $A \wedge B$ is true, if and only if both A and B are true.

A	B	$A \wedge B$
false	false	false
false	true	false
true	false	false
true	true	true

Operational Interpretation

```
public final class And implements Formula
{
  private Formula formula0; private Formula formula1;

  public boolean eval() throws EvalException
  {
    if (formula0.eval())
      if (formula1.eval())
        return true;
      else
        return false;
    else
      return false;
  }
}
```


Conjunctive Laws

Proposition:

1. Conjunction is commutative: for all formulas A and B,

$$A \wedge B \text{ iff } B \wedge A.$$

2. Conjunction is associative: for all formulas A, B, and C,

$$A \wedge (B \wedge C) \text{ iff } (A \wedge B) \wedge C.$$

Proof

Proof: We prove associativity. Take arbitrary formulas A, B, and C. We show that $A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C$ have same truth value.

A	B	C	$B \wedge C$	$A \wedge (B \wedge C)$	$A \wedge B$	$(A \wedge B) \wedge C$
false	false	false	false	false	false	false
false	false	true	false	false	false	false
false	true	false	false	false	false	false
false	true	true	true	false	false	false
true	false	false	false	false	false	false
true	false	true	false	false	false	false
true	true	false	false	false	true	false
true	true	true	true	true	true	true

Notation

Because of associativity, setting of parentheses in nestings of conjunctive formulas do not matter.

- $A \wedge B \wedge C$
  - $A \wedge (B \wedge C)$
  - $(A \wedge B) \wedge C$
- $A_0 \wedge A_1 \wedge \ldots \wedge A_{n-1}$
- and$(A_0, A_1, \ldots, A_{n-1})$

Disjunctions

$$(A \vee B)$$

- Other syntactic forms:
  - $A + B$, $A \mid B$, A || B;
  - "A or B";
  - `or(A, B)`.
- Semantics: $A \vee B$ is false, if and only if both A and B are false.

A	B	$A \vee B$
false	false	false
false	true	true
true	false	true
true	true	true

Operational Interpretation

```
public final class Or implements Formula
{
  private Formula formula0; private Formula formula1;

  public boolean eval() throws EvalException
  {
    if (formula0.eval())
      return true;
    else
      if (formula1.eval())
        return true;
      else
        return false;
  }
}
```


Disjunctive Laws

Proposition:

1. Disjunction is commutative: for all formulas A and B

$$A \vee B \text{ iff } B \vee A.$$

2. Disjunction is associative: for all formulas A, B, and C, we have

$$A \vee (B \vee C) \text{ iff } (A \vee B) \vee C.$$

Proof: by truth tables.

De Morgan's Laws

Proposition: For all formulas A and B, the following holds:

$$\neg(A \wedge B) \text{ iff } \neg A \vee \neg B,$$
$$\neg(A \vee B) \text{ iff } \neg A \wedge \neg B.$$

Proof: by truth table.

Consequence: For all formulas A and B, we have:

$$A \vee B \text{ iff } \neg(\neg A \wedge \neg B).$$

Proof: by inversion of negation.

Disjunction can thus be defined by conjunction and negation.

Notation

Because of associativity, setting of parentheses in nestings of disjunctive formulas do not matter.

- $A \lor B \lor C$
  - $A \lor (B \lor C)$
  - $(A \lor B) \lor C$
- $A_0 \lor A_1 \lor \ldots \lor A_{n-1}$
- $\texttt{or(}A_0\texttt{, }A_1\texttt{, ..., }A_{n-1}\texttt{)}$

Exclusive Disjunction

$$(A \text{ xor } B)$$

- Read: either A or B.
- Semantics: $A \text{ xor } B$ is true, iff exactly one of A or B is true.

A	B	$A \text{ xor } B$
false	false	false
false	true	true
true	false	true
true	true	false

- Exclusive disjunction can thus be defined by other connectives.

$$(A \text{ xor } B) \text{ iff } (A \land \neg B) \lor (\neg A \land B).$$

Implication

$$(A \Rightarrow B)$$

- Other syntactic forms:
  - "A implies B"; "B follows from A"
  - "if A, then B"; "B, (only) if A"
  - "A is sufficient for B"; "B is necessary for A"
  - $\texttt{implies(}A\texttt{, }B\texttt{)}$.
- Semantics: $A \Rightarrow B$ is false if and only if A is true and B is false.

A	B	$A \Rightarrow B$
false	false	true
false	true	true
true	false	false
true	true	true

Operational Interpretation

```
public final class Implies implements Formula
{
  private Formula formula0; private Formula formula1;

  public boolean eval() throws EvalException
  {
    if (formula0.eval())
      if (formula1.eval())
        return true;
      else
        return false;
    else
      return true;
  }
}
```

Implicative Laws

Proposition: For all formulas A and B, we have

$$\begin{aligned} A \Rightarrow B &\text{ iff } \neg A \vee B \\ A \Rightarrow B &\text{ iff } \neg B \Rightarrow \neg A \\ \neg(A \Rightarrow B) &\text{ iff } A \wedge \neg B. \end{aligned}$$

Implication can thus be defined by negation and disjunction.

Proposition: Implication is not associative.
Proof: by construction of a counterexample:

- Meaning of $\mathrm{F} \Rightarrow (\mathrm{F} \Rightarrow \mathrm{F})$ is true.
- Meaning of $(\mathrm{F} \Rightarrow \mathrm{F}) \Rightarrow \mathrm{F}$ is false.

Equivalences

$$(A \Leftrightarrow B)$$

- Other syntactic forms:
  - $A = B$, $A \sim B$;
  - "A and B are equivalent", "A if and only if B", "A iff B";
  - "A is necessary and sufficient for B";
  - equiv(A, B).
- Semantics: $A \Leftrightarrow B$ is true iff A and B have the same truth value.

A	B	$A \Leftrightarrow B$
false	false	true
false	true	false
true	false	false
true	true	true

Operational Interpretation

```
public final class Equiv implements Formula
{
  private Formula formula0; private Formula formula1;
  ...

  public boolean eval() throws EvalException
  {
    return (formula0.eval() == formula1.eval());
  }
}
```


Equivalence Laws

Proposition: For all formulas A and B, the following holds:

$$\begin{aligned} A \Leftrightarrow B &\text{ iff } B \Leftrightarrow A \\ A \Leftrightarrow B &\text{ iff } (A \wedge B) \vee (\neg A \wedge \neg B) \\ A \Leftrightarrow B &\text{ iff } (A \Rightarrow B) \wedge (B \Rightarrow A) \end{aligned}$$

Equivalence can thus be defined by implication and conjunction.

Proposition: Equivalence is not associative.
Proof: by construction of a counterexample.

Propositional Logic

- Semantics:

A	B	F	T	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
false	false	false	true	true	false	false	true	true
false	true	false	true	true	false	true	true	false
true	false	false	true	false	false	true	false	false
true	true	false	true	false	true	true	true	true

- Relationships:

$$A \vee B \text{ iff } \neg(\neg A \wedge \neg B)$$
$$A \Rightarrow B \text{ iff } \neg A \vee B$$
$$A \Leftrightarrow B \text{ iff } (A \Rightarrow B) \wedge (B \Rightarrow A)$$

All connectives can be ultimately reduced to negation and conjunction.

Logic Evaluator

```
formula or(true, false);
> true.
formula or(false, not(false));
> true.
formula or(true, true);
> true.

formula or(not(true), and(true, false));
```

```
formula implies(not(true), true);
> true.
formula implies(and(true, false), false);
> true.

formula implies(or(true, false), not(false));
```

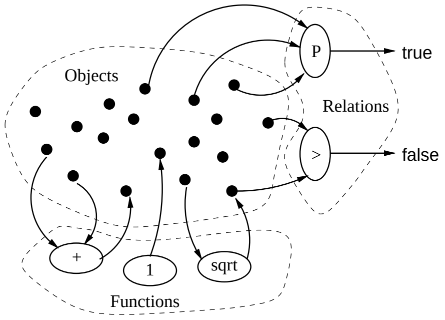

Predicate Logic

Predicate Logic

Superset of propositional logic that describes values and properties of a domain consisting of:

1. A collection of values (objects, entities).
2. A collection of functions (mappings):
   - Takes a certain number of values (the arguments);
   - Returns a value (the result);
   - Number of arguments is the arity of the function;
   - Zero-arity functions are called (object) constants.
3. A collection of predicates (relations, properties, attributes):
   - Takes a certain number of values (the arguments);
   - Returns a truth value;
   - Number of arguments is the arity of the predicate.

Domain



An abstract image of the real world.

Example

- The set of natural numbers
  - constants “zero” and “one”, binary functions “addition”, “multiplication”;
  - binary predicate “is less than’;
- The people in Austria
  - functions “mother of” and “father of”;
  - unary predicates “is male” and “is female”; ternary predicate “are parents of”;
- The set of Java values of types `int` and `int[]`
  - function “indexed array access”;
  - predicate “is sorted”.

Predicate Logic

Extension of propositional logic by

- Terms
  - Syntactic phrases denoting objects.
  - Variables.
  - Function applications.
- Atomic formulas
  - Formulas using terms to express facts about particular objects of a domain.
- Quantified formulas
  - Formulas expressing facts about all objects of a domain.

Terms

Definition: Term

A syntactic phrase whose semantics is a value (of the domain being considered).

$$\boxed{\text{Term} \xrightarrow{\text{Meaning}} \text{Domain Value}}$$

Operational Interpretation

```
public interface Term
{
  Value eval() throws EvalException;
}

Term term = ...;
Value meaning = term.eval();
```

Evaluation of a term returns a domain value (normally).

Variables and Assignments

Definition: Variable.

- A name that may represent any value of the domain.
- First-order: must not represent predicates or functions.

Example: x, y, ...

Definition: (Variable) assignment.

- A mapping of all variables to domain values.

Example: $[x \mapsto 1, y \mapsto 2, \ldots]$

Operational Interpretation

```
public final class Variable implements Term
{
  private String variable;

  public Value eval() throws EvalException
  {
    Value value = Context.get(variable);
    if (value == null) throw
      new EvalException("no variable " + variable + " in context");
    return value;
  }
}
```


Function Constants

Definition: Function constants.

- A name that may represent some function.
- The arity of the constant determines the arity of the functions it may stand for.

Examples:

- 0, 1, 3.14, “Wolfgang Schreiner”, “Austria”, π (object constants);
- $|\ |$, sin(), $\sqrt{\ }$, “mother of” (unary constants);
- $+$, $\circ$, $[\ ]$ (binary constants);

Syntax of Terms

Proposition:

1. Every variable is a term.
2. If fc is a function constant of arity n and $t_0, \ldots, t_{n-1}$ are terms,

$$fc(t_0, \ldots, t_{n-1})$$

is a term, called an elementary term or function application.

Example:

- 0, "Wolfgang Schreiner", π (constant terms);
- $\sqrt{2}$, sin(x), sum(2, s), "the mother of Thomas" (prefix terms);
- $|1|$, $a \circ b$, $a[i]$, $1 + 2$ (infix terms);

Natural Language Terms

"the roof of the house of her father"

$$\mathrm{roof}(\mathrm{house}(\mathrm{father}(\mathrm{she})))$$

- Variables:
  - she
- Unary function constants:
  - father
  - house
  - roof

Semantics of Terms

Definition: The meaning of a term t under an assignment a is:

- If t is a variable, then its meaning is the value to which the variable is mapped by the assignment.
- If t is an elementary term $fc(t_0, \ldots, t_{n-1})$ then its meaning is the result of the application of the function f denoted by the function constant fc to the values of the terms t_i for the given assignment.
  1. Determine values of the terms t_i under a.
  2. Determine function f denoted by constant fc.
  3. Apply f to values and get domain value.

Examples

Term $x + (y + 0)$.

- Domain "natural numbers".
  - Object constant 0 interpreted as "zero".
  - Binary function constant $+$ interpreted as addition.
  - Assignment $[x \mapsto$ "one", $y \mapsto$ "two"$]$. gives natural number "three" as meaning.
  - Assignment $[x \mapsto$ "one", $y \mapsto$ "zero"$]$ gives natural number "one" as meaning.
- Domain "character strings".
  - Object constant 0 interpreted as "empty string".
  - Binary function constant $+$ interpreted as string concatenation.
  - Assignment $[x \mapsto \text{“}hi,\text{ ”}, y \mapsto \text{“}babe\text{”}]$ gives string "hi, babe" as meaning.

Semantics of a term depends on domain and variable assignment.

Operational Interpretation

```
public final class Application implements Term
{
  private String name; private Term[] arguments;

  public Value eval() throws EvalException
  {
    Function function = Model.getFunction(name, arguments.length);
    if (function == null) throw new EvalException("unknown function");
    Value[] values = new Value[arguments.length];
    for (int i=0; i< values.length; i++)
      values[i] = arguments[i].eval();
    return function.apply(values);
  }
}
```


Predicate Constants

Definition: Predicate Constant.

- A name that may represent some predicate.
- The arity of the constant determines the arity of the predicates it may stand for.

Example:

- "is positive" (unary predicate constant);
- $\leq$, $|$ (binary predicate constants);
- "is father of" (binary predicate constant);
- "is a child of ... and of ..." (ternary predicate constant).

Syntax of Atomic Formulas

Definition: Atomic Formula

$$pc(t_0, \ldots, t_{n-1})$$

where pc is a predicate constant of arity n and $t_0, \ldots, t_{n-1}$ are terms.

Example:

- "1 is positive";
- $2 \leq \sqrt{x+3}$;
- $2|5$;
- "Thomas is the father of Susanne";
- "Susanne is a child of Thomas and of Birgit".

Natural Language Atomic Formulas

"Bill Clinton is a better president than his predecessor".

$$\text{isBetterPresident}(\text{BillClinton}, \text{Predecessor}(\text{BillClinton}))$$

- Function constants:
  - Object constant "Bill Clinton".
  - Unary constant "Predecessor".
- Predicate constants:
  - Binary constant "isBetterPresident".

Semantics of Atomic Formulas

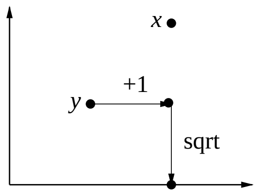
Definition: The meaning of an atomic formula $pc(t_0, \ldots, t_{n-1})$ under an assignment a

- The truth value of the predicate denoted by the predicate constant pc for the values of the terms t_i under the given assignment.
  1. Determine values of the terms t_i under a.
  2. Determine predicate p denoted by constant pc.
  3. Apply p to values and get truth value.

Example

Formula $x \leq \sqrt{y+1}$.

- Domain of real numbers.
  - Function and object constants interpreted as usual.
  - Assignment $[x \mapsto 3, y \mapsto 8]$ gives meaning true;
  - Assignment $[x \mapsto 4, y \mapsto 3]$ gives false.
- Domain of points in a plane.
  - $\leq$interpreted as "is not farther from the zero point than";



Operational Interpretation

```
public final class Atomic implements Formula
{
  private String name; private Term[] arguments;

  public boolean eval() throws EvalException
  {
    Predicate predicate = Model.getPredicate(name,arguments.length);
    if (predicate == null) throw new EvalException("unknown predicate");
    Value[] values = new Value[arguments.length];
    for (int i=0; i<values.length; i++)
      values[i] = arguments[i].eval();
    return predicate.apply(values);
  }
}
```


Equality

Predicate constant with the following properties, for all x, y, and z:

- Reflexivity: $x = x$;
- Symmetry: $x = y \Rightarrow y = x$;
- Transitivity: $(x = y \wedge y = z) \Rightarrow x = z$.
- Equality Axiom (every function constant f or predicate constant p)

$$x = y \;\Rightarrow\; f(\ldots, x, \ldots) = f(\ldots, y, \ldots);$$
$$x = y \;\Rightarrow\; p(\ldots, x, \ldots) \Leftrightarrow p(\ldots, y, \ldots).$$

First Order Predicate Logic with Equality.

Quantified Formulas

Definition: Quantifier

- A syntactic operator that combines a variable and a phrase to form a new phrase.
- The meaning of the new phrase does not depend on the value of the variable in the given assignment.
- We call this variable bound by the quantifier.
- If all variables in a phrase are bound, the phrase is called closed.

Example: Formula $x < 1$.

- x is free (not bound) in this formula.

Universal Quantification

$$(\forall x : A)$$

- Variable x bound by universal quantifier $\forall$.
- Other syntactic forms:
  - $\forall_x A$; $\bigwedge_x A$;
  - "for all x we have A"; "every x has A"; "A, for all x";
  - `forall(`x`:`A`)`.
- Semantics: $(\forall x : A)$ is true, iff A is true for every value of x.

  A is true in every extension of the given assignment where x is mapped to a domain value.

Operational Interpretation

```
public final class ForAll implements Formula
{
  private String variable; private Term domain; private Formula formula;

  public boolean eval() throws EvalException
  {
    Iterator iterator = Model.iterator(domain);
    while (iterator.hasNext()) {
      Context.begin(variable, iterator.next());
      boolean result = formula.eval();
      Context.end();
      if (!result) return false; }
    return true;
  }
}
```


Existential Quantification

$$(\exists x : A)$$

- Variable x bound by existential quantifier $\exists$.
- Other syntactic forms:
  - $\exists_x A$; $\bigvee_x A$;
  - "there exists x with A"; "there is some x with A"; "some x has A"; "A, for some x";
  - `exists(`x`:`A`)`.
- Semantics: $(\exists x : A)$ is true iff A is true for some value of x.

  A is true in some extension of the given assignment where x is mapped to a domain value.

Operational Interpretation

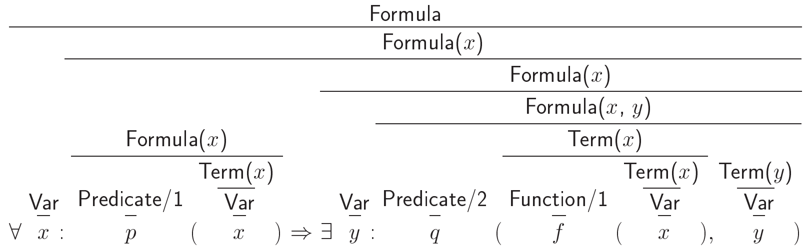
```
public final class Exists implements Formula
{
  private String variable; private Term domain; private Formula formula;

  public boolean eval() throws EvalException
  {
    Iterator iterator = Model.iterator(domain);
    while (iterator.hasNext()) {
      Context.begin(variable, iterator.next());
      boolean result = formula.eval();
      Context.end();
      if (result) return true; }
    return false;
  }
}
```


Syntax Analysis

$$\forall x : p(x) \Rightarrow \exists y : q(f(x), y)$$

$$\forall x : (p(x) \Rightarrow \exists y : q(f(x), y)).$$



Examples

- $(\forall x : y \leq x)$
  - x is bound, y is free; the formula is therefore not closed.
  - true in assignment $[y \mapsto 0]$ over the domain of natural numbers with the usual interpretation of '$\leq$', because $0 \leq x$ is true for every natural number x:

$$0 \leq 0, 0 \leq 1, 0 \leq 2, \ldots$$

- $(\exists x : x|15)$
  - x is bound; the formula is closed.
  - true for every assignment over the natural numbers with '$|$' interpreted as 'divides', because $x|15$ is true for some natural number x (e.g. for 3):

$$3|15.$$

Bound Variables

The meaning of

$$\forall x : \exists y : x * y \leq z$$

is the same as the meaning of

$$\forall y : \exists w : y * w \leq z$$

but it is not the same as the meaning of

$$\forall x : \exists y : x * y \leq w.$$

Renaming a bound variable does not change the meaning of a phrase.

Closed Formulas

Do not just write

$$x + y = y + x$$

if you wish to say, e.g.

$$x + y = y + x, \text{ for every } x \text{ and } y,$$

i.e.,

$$\forall x, y : x + y = y + x.$$

Bind variables by quantifiers to avoid misinterpretations.

Abbreviations

- $\forall x : \forall y : A$

$$\forall x, y : A$$

- $\exists x : \exists y : A$

$$\exists x, y : A$$

Multiple quantifiers of the same type can be merged.

Quantifier Patterns

- $\forall x : A_x \Rightarrow B$

$$\forall A_x : B$$

- $\exists x : A_x \wedge B$

$$\exists A_x : B.$$

Example:

- $\forall x : \exists 0 \leq y \leq x : |x - 2 * y| \leq 1$
- $\forall x : (\exists y : (0 \leq y) \wedge (y \leq x) \wedge (|x - 2 * y| \leq 1))$

Bound variable has to be deduced from the context.

Examples

- "All Ferraris are red"
  - "It holds for all objects that, if the object is a Ferrari, then the object is red".
  - $\forall x : \text{isFerrari}(x) \Rightarrow \text{isRed}(x)$.
- "Some Ferraris are black"
  - "There exists an object such that the object is a Ferrari and the object is black"
  - $\exists x : \text{isFerrari}(x) \wedge \text{isBlack}(x)$.
- "Everybody has exactly one father".
  - "For every person x, there is one person y such that y is the father of x and every person z who is father of x is identical to y".
  - $(\forall x : (\exists y : \text{isfather}(y, x) \wedge (\forall z : \text{isfather}(z, x) \Rightarrow z = y)))$.
  - $\forall x : \exists y : \text{isfather}(y, x) \wedge \forall z : \text{isfather}(z, x) \Rightarrow z = y$.

De Morgan's Laws

For every variable x and formula A, we have

- $$\neg\forall x : A \text{ iff } \exists x : \neg A$$

  "Not every x satisfies A" equals "there exists some x with $\neg A$".

- $$\neg\exists x : A \text{ iff } \forall x : \neg A$$

  "There exists some x with A" equals "not every x satisfies $\neg A$".

Universal and existential quantification are dual concepts.

Logic Evaluator

```
option universe = nat(0, 100);
> universe of discourse set.
formula forall(x: <=(x, 1000));
> true.
formula exists(x: =(*(x, 2), 56));
> true.
formula exists(x: =(*(x, 2), 49));
> false.
formula forall(x: exists(y:
  or(=(*(2, y), x),
     =(*(2, y), +(x, 1)))));
> true.

formula exists(x: forall(y: <=(y, x)));
```


Local Definitions

$$(\textbf{let } x = T : P) \; (P \textbf{ where } x = T)$$

- Term T, any phrase (term or formula) P.
- Variable x bound by "block quantifier" **let**/**where**.
- Alternative forms:
  - Let x be T. Then we have A.
  - Let x be T in P; A, where $x = T$.
  - `let(`x `=` T`:` A`)`
- Semantics: identical to the semantics of P with x replaced by T.

  Semantics of P in an extension of the given assignment with x mapped to value of T.

Operational Interpretation

```
public final class LetTerm implements Term
{
  private String variable; private Term term; private Term body;

  public Value eval() throws EvalException
  {
    Context.begin(variable, term.eval());
    Value result = body.eval();
    Context.end();
    return result;
  }
}
```

Example

Take the domain of natural numbers and assignment $[x \mapsto 1]$.

- The proposition
$$\textbf{let } y = 0 : x \leq y$$
is false because $1 \leq 0$ does not hold.
- The function f defined as
$$f(x) := s * x + s \textbf{ where } s = x + 1$$
has the same meaning as if it were defined as
$$f(x) := (x + 1) * x + (x + 1).$$

Example

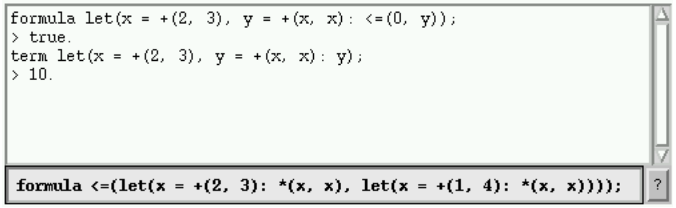
- The formula
$$x|y \land \exists z : y|z \textbf{ where } y = 2x$$
is equivalent to $x|2x \land \exists z : 2x|z$.
- The formula
$$x|y \land \exists y : y|x \textbf{ where } y = 2x$$
is equivalent to $x|2x \land \exists y : y|x$ but not to $x|2x \land \exists y : 2x|x$.

Abbreviation

Instead of writing
$$\textbf{let } x = T_0 : \textbf{let } y = T_1 : A$$
we usually write
$$\textbf{let } x = T_0, y = T_1 : A$$
or correspondingly
$$A \textbf{ where } x = T_0, y = T_1.$$

Multiple bindings can be merged.

Logic Evaluator



Example

Example

Problem Statement:

Write a program that takes a number and returns the next prime number.

Program Specification

- Input: n such that $isNumber(n)$;
- Output: p such that $isNextPrime(n,p)$.

Program Specification

- Input condition: $isNumber$

  Unary predicate that describes well-formed input n.
- Output condition: $isNextPrime$

  Binary predicate that describes how the output p is related to the input n.

Specification of an explicit construction problem.

Input and Output Conditions

Definition: n is a number.

$$isNumber(n) :\Leftrightarrow n \in \mathbb{N}$$

Definition: p is the next prime number after n.

$$\begin{aligned} & isNextPrime(n,p) :\Leftrightarrow \\ & \quad isNumber(p) \wedge \\ & \quad isPrime(p) \wedge \\ & \quad isNextP(n,p). \end{aligned}$$

Output Condition

Definition: p is a prime number.

$$\begin{aligned}&isPrime(p) :\Leftrightarrow\\&\quad 1 < p \wedge\\&\quad \neg(\exists 1 < n < p : n|p).\end{aligned}$$

A bit more elegant:

$$\begin{aligned}&isPrime(p) :\Leftrightarrow\\&\quad 1 < p \wedge\\&\quad \forall 1 < n < p : n \not| \, p.\end{aligned}$$

Output Condition

Definition: p is the next such number after n.

Option 1: if n is prime, then $isNextP(n, n)$.

$$\begin{aligned}&isNextP(n, p) :\Leftrightarrow\\&\quad n \leq p \wedge\\&\quad \neg(\exists n \leq q < p : isPrime(q)).\end{aligned}$$

Option 2: even if n is prime, $\neg isNextP(n, n)$.

$$\begin{aligned}&isNextP(n, p) :\Leftrightarrow\\&\quad n < p \wedge\\&\quad \neg(\exists n < q < p : isPrime(q)).\end{aligned}$$

Complete Specification

- **Input:** $n \in \mathbb{N}$;
- **Output:** $p \in \mathbb{N}$ such that $isNextPrime(n, p)$.

$$\begin{aligned}&isNextPrime(n, p) :\Leftrightarrow\\&\quad n \leq p \wedge isPrime(p) \wedge\\&\quad \neg(\exists n \leq q < p : isPrime(q))\end{aligned}$$

$$\begin{aligned}&isPrime(p) :\Leftrightarrow\\&\quad 1 < p \wedge \forall 1 < n < p : n \not| \, p\end{aligned}$$

$$n|m :\Leftrightarrow \exists p : n * p = m.$$

Logic Evaluator

```
pred <(m, n) <=> and(<=(m, n), not(=(m, n)));

pred divides(n, m) <=> exists(p in nat(1, m): =(*(n, p), m));

pred isPrime(p) <=>
  and(<(1, p),
      forall(n in nat(2, -(p, 1)): not(divides(n, p))));

pred isNextPrime(n, p) <=>
  and(<=(n, p), isPrime(p),
      not(exists(q in nat(n, -(p, 1)): isPrime(q))));

fun program(n) =
  such(p in nat(n, *(2, n)): isNextPrime(n, p), p);
```

Logic Evaluator

```
pred <(m, n) <=> and(<=(m, n), not(=(m, n)));
> predicate </2.
pred divides(n, m) <=> exists(p in nat(1, m): =(*(n, p), m));
> predicate divides/2.
pred isPrime(p) <=>
  and(<(1, p),
      forall(n in nat(2, -(p, 1)): not(divides(n, p))));
> predicate isPrime/1.
pred isNextPrime(n, p) <=>
  and(<=(n, p), isPrime(p),
      not(exists(q in nat(n, -(p, 1)): isPrime(q))));
> predicate isNextPrime/2.
fun program(n) = such(p in nat(n, *(2, n)): isNextPrime(n, p), p);
> function program/1.
term program(14);
> 17.
term program(50);
> 53.
```

```
term program(90);
```


Summary

- Logical connectives.
  - Truth tables.
- Domains, constants, interpretations.
- Terms
  - Variables and assignments.
  - Function applications.
- Atomic formulas.
- Quantified formulas.
  - Free and bound variables; closed formulas.
  - Universal and existential quantification.
  - Local definitions.