S_5 acts on $\{0,1\}^{\binom{[5]}{2}}$ in the usual way. We want to count the number of orbits of this action.

We first list all cycle types in S_5:

( )( )( )( )( )	1
(,)(,)( )	15
(,)( )( )( )	10
(, ,)( )( )	20
(, ,)(,)	20
(, , ,)( )	30
(, , , ,)	24

We now use the following formula to count the ~~different cycle~~ number of permutations with a given cycle type in S_n:

$$\frac{n!}{\lambda_1^{m_1} \cdots \lambda_k^{m_k} m_1! m_2! \cdots m_k}$$

The formula counts how many permutations there are in S_n with cycle type

$$\underbrace{(\underbrace{\quad}_{\lambda_1})(\underbrace{\quad}_{\lambda_1})\cdots(\underbrace{\quad}_{\lambda_1})}_{m_1 \text{ cycles}}\underbrace{(\underbrace{\quad}_{\lambda_2})(\underbrace{\quad}_{\lambda_2})\cdots(\underbrace{\quad}_{\lambda_2})}_{m_2 \text{ cycles}}\cdots$$

Ex: (,)(,)( )

$\lambda_1 = 1 \quad m_1 = 1$

$\lambda_2 = 2 \quad m_2 = 2$

$$\frac{5!}{1^1 2^2 1! 2!} = 15$$

$(\, , \,)(\,)(\,)(\,)$ $\lambda_1 = 1$ $m_1 = 3$

$\lambda_1 = 2$ $m_2 = 1$

$$\frac{5!}{1^3 \, 2^1 \, 3! \, 1!} = \frac{120}{2 \cdot 6} = 10$$

$(\, , \, , \,)(\,)(\,)$ $\lambda_1 = 1$ $m_1 = 2$

$\lambda_2 = 3$ $m_2 = 1$

$$\frac{5!}{1^2 \, 3^1 \cdot 2! \, 1!} = \frac{120}{1 \cdot 3 \cdot 2 \cdot 1} = 20$$

$(\, , \, , \,)(\, , \,)$ $\lambda_1 = 2$ $m_1 = 1$

$\lambda_2 = 3$ $m_2 = 1$

$$\frac{5!}{2^1 \, 3^1 \, 1! \, 1!} = \frac{120}{6} = 20$$

$(\, , \, , \, , \,)(\,)$ $\lambda_1 = 1$ $m_1 = 1$

$\lambda_2 = 4$ $m_2 = 1$

$$\frac{5!}{1^1 \, 4^1 \, 1! \, 1!} = \frac{120}{4} = 30$$

$(\, , \, , \, , \, , \,)$ $\lambda_1 = 5$ $m_1 = 1$

$$\frac{5!}{5^1 \cdot 1!} = \frac{120}{5} = 24$$

Now we apply Burnside's lemma,

$$|G\backslash X| = \frac{1}{|S_5|}\sum_{g \in S_5} |X_g|$$
$$= |S_5 \backslash\backslash \{0,1\}^{\binom{[5]}{2}}|$$

We first consider $g = (1)(2)(3)(4)(5)$

$$|X_g| = |X| = 2^{\binom{5}{2}} = 2^{10}$$

Now we consider $g = (1,2)(3,4)(5)$

$$\binom{[5]}{2} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},$$
$$\{3,4\},\{3,5\},\{4,5\}\}$$

We now consider the usual action of S_5 on $\binom{[5]}{2}$: $g\{i,j\} \mapsto \{g(i), g(j)\}$

$\{1,2\} \xrightarrow{g} \{1,2\} = \{1,2\}$ (loop g)

$\{1,3\} \xrightarrow{g} \{2,4\} \xrightarrow{g} \{1,3\} = \{1,3\} \xrightarrow{g} \{2,4\}$ (and back by g)

$\{1,4\} \xrightarrow{g} \{2,3\}$ (and back by g)

$\{1,5\} \xrightarrow{g} \{2,5\}$ (and back by g)

$\{3,4\}$ (loop g)

$\{3,5\} \xrightarrow{g} \{4,5\}$ (and back by g)

We can summarize this:

$$(\{1,2\})(\{1,3\},\{2,4\})(\{1,4\},\{2,3\})(\{1,5\},\{2,5\})$$
$$(\{3,4\})(\{3,5\},\{4,5\})$$

For displaying an arbitrary $f \in \{0,1\}^{\binom{[5]}{2}}$ we use

$$f = \left(\begin{array}{c|cc|cc|cc|c|cc} \{1,2\} & \{1,3\} & \{2,4\} & \{1,4\} & \{2,3\} & \{1,5\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right)$$

$$gf = \left(\begin{array}{c|cc|cc|cc|c|cc} g\{1,2\} & g\{1,3\} & g\{2,4\} & g\{1,4\} & g\{2,3\} & g\{1,5\} & g\{2,5\} & g\{3,4\} & g\{3,5\} & g\{4,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right)$$

$$= \left(\begin{array}{c|cc|cc|cc|c|cc} \{1,2\} & \{2,4\} & \{1,3\} & \{2,3\} & \{1,4\} & \{2,5\} & \{1,5\} & \{3,4\} & \{4,5\} & \{3,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right)$$

$$= \left(\begin{array}{c|cc|cc|cc|c|cc} \{1,2\} & \{1,3\} & \{2,4\} & \{1,4\} & \{2,3\} & \{1,5\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \\ x_1 & x_3 & x_2 & x_5 & x_4 & x_7 & x_6 & x_8 & x_{10} & x_9 \end{array}\right)$$

$$gf = f \iff x_2 = x_3 \quad x_5 = x_4 \quad x_6 = x_7 \quad x_{10} = x_9$$

$X = \{0,1\}^{\binom{[5]}{2}}$

$X_g = \{ f \in X \mid gf = f \}$

$= \{ f \mid x_2 = x_3,\ x_5 = x_4,\ x_6 = x_7,\ x_9 = x_{10},\ x_i \in \{0,1\} \}$, $f \in X$

x_1	x_2, x_3	x_4, x_5	x_6, x_7	x_8	x_9, x_{10}
0	0	0	0	0	0
0	0	0	0	0	1
0	0	0	0	1	0

⋮

2^6 possibilities

Hence $|X_g| = 2^6$

$g = (1,2)(3)(4)(5)$

$\{1,2\} \xrightarrow{g} \{1,2\}$ $\quad \{1,3\} \xrightarrow{g} \{2,3\} \xrightarrow{g} \{1,3\}$ $\quad \{1,4\} \xrightarrow{g} \{2,4\} \xrightarrow{g} \{1,4\}$

$\{1,5\} \xrightarrow{g} \{2,5\} \xrightarrow{g} \{1,5\}$ $\quad \{3,5\} \xrightarrow{g} \{3,5\}$ $\quad \{3,4\} \xrightarrow{g} \{3,4\}$ $\quad \{4,5\} \xrightarrow{g} \{4,5\}$

$(\{1,2\})(\{3,5\})(\{3,4\})(\{4,5\})(\{1,3\},\{2,3\})(\{1,4\},\{2,4\})(\{1,5\},\{2,5\})$

$$f = \begin{pmatrix} \{1,2\} & \{3,5\} & \{3,4\} & \{4,5\} & \{1,3\} & \{2,3\} & \{1,4\} & \{2,4\} & \{1,5\} & \{2,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{pmatrix}$$

$$gf = \begin{pmatrix} \{1,2\} & \{3,5\} & \{3,4\} & \{4,5\} & \{1,3\} & \{2,3\} & \{1,4\} & \{2,4\} & \{1,5\} & \{2,5\} \\ x_1 & x_2 & x_3 & x_4 & x_6 & x_5 & x_8 & x_7 & x_{10} & x_9 \end{pmatrix}$$

$$X_g = \left\{ f \;\middle|\; f \in X,\ \begin{matrix} x_5 = x_6 \\ x_7 = x_8 \\ x_9 = x_{10} \end{matrix} \quad x_i \in \{0,1\} \right\}$$

$$|X_g| = 2^7$$

$g = (1,2,3)(4)(5)$

$\{1,2\} \xrightarrow{g} \{2,3\} \xrightarrow{g} \{1,3\} \xrightarrow{g} \{1,2\}$

$\{1,4\} \xrightarrow{g} \{2,4\} \xrightarrow{g} \{3,4\} \xrightarrow{g} \{1,4\}$

$\{1,5\} \xrightarrow{g} \{2,5\} \xrightarrow{g} \{3,5\} \xrightarrow{g} \{1,5\}$

$\{4,5\} \xrightarrow{g} \{4,5\}$

$$(\{1,2\},\{2,3\},\{1,3\})(\{1,4\},\{2,4\},\{3,4\})(\{1,5\},\{2,5\},\{3,5\})(\{4,5\})$$

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$$f = \left(\begin{array}{ccc|ccc|ccc|c} \{1,2\} & \{2,3\} & \{1,3\} & \{1,4\} & \{2,4\} & \{3,4\} & \{1,5\} & \{2,5\} & \{3,5\} & \{4,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right)$$

$$gf = \left(\begin{array}{ccc|ccc|ccc|c} \{2,3\} & \{1,3\} & \{1,2\} & \{2,4\} & \{3,4\} & \{1,4\} & \{2,5\} & \{3,5\} & \{1,5\} & \{4,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc|ccc|c} \{1,2\} & \{2,3\} & \{1,3\} & \{1,4\} & \{2,4\} & \{3,4\} & \{1,5\} & \{2,5\} & \{3,5\} & \{4,5\} \\ x_3 & x_1 & x_2 & x_6 & x_4 & x_5 & x_9 & x_7 & x_8 & x_{10} \end{array}\right)$$

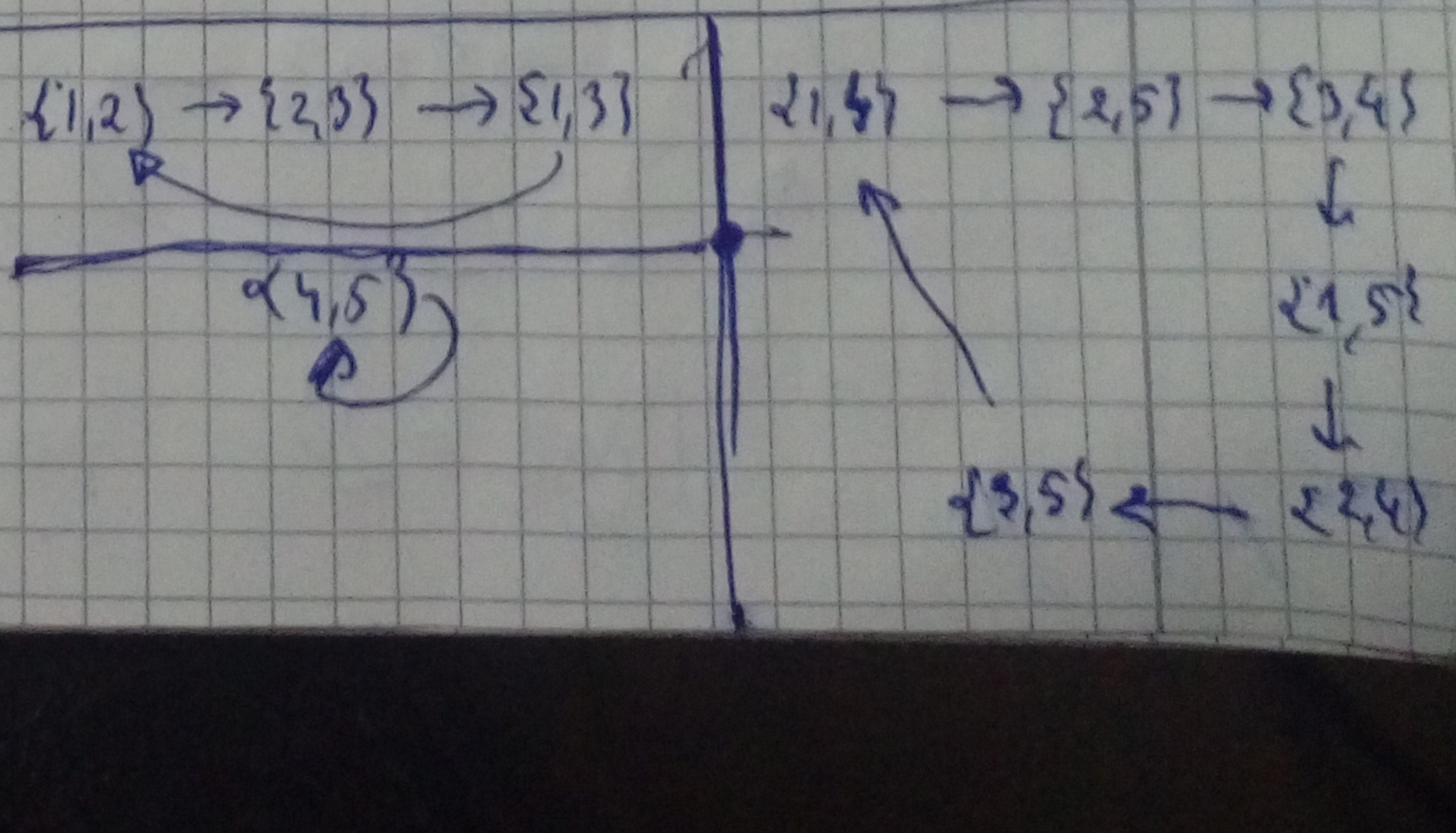
$gf = f \iff$

$x_1 = x_2$	$x_4 = x_5$	$x_7 = x_8$
$x_2 = x_3$	$x_5 = x_6$	$x_8 = x_9$
$x_3 = x_1$	$x_6 = x_4$	$x_9 = x_7$

$\iff$ $x_1 = x_2 = x_3$
$x_4 = x_5 = x_6$
$x_7 = x_8 = x_9$

$$|X_g| = \left| \left\{ f \mid f \in X,\ \begin{array}{l} x_1 = x_2 = x_3 \\ x_4 = x_5 = x_6 \\ x_7 = x_8 = x_9 \end{array},\ x_i \in \{0,1\} \right\} \right| = 2^4$$

$g = (1,2,3)(4,5)$



$$(\{1,2\},\{2,3\},\{1,3\})(\{1,4\},\{2,5\},\{3,4\},\{1,5\},\{2,4\},\{3,5\})(\{4,5\})$$

$$f = \left(\begin{array}{ccc|cccccc|c} \{1,2\} & \{2,3\} & \{1,3\} & \{1,4\} & \{2,5\} & \{3,4\} & \{1,5\} & \{2,4\} & \{3,5\} & \{4,5\} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right)$$

$$gf = \left(\begin{array}{ccc|cccccc|c} \{1,2\} & \{2,3\} & \{1,3\} & \{1,4\} & \{2,5\} & \{3,4\} & \{1,5\} & \{2,4\} & \{3,5\} & \{4,5\} \\ x_3 & x_1 & x_2 & x_9 & x_4 & x_5 & x_6 & x_7 & x_8 & x_{10} \end{array}\right)$$

$$gf = f \iff \begin{array}{l} x_1 = x_2 = x_3 \\ x_4 = x_5 = x_6 = x_7 = x_8 = x_9 \end{array}$$

$$|X_g| = 2^3$$

$$g = (1,2,3,4)(5)$$

$$\{1,2\} \to \{2,3\} \to \{3,4\} \to \{4,1\} \to \{1,2\}$$

$$\{1,3\} \to \{2,4\} \to \{1,3\}$$

$$\{1,5\} \to \{2,5\} \to \{3,5\} \to \{4,5\} \to \{1,5\}$$

$$|X_g| = 2^3$$

$g = (1,2,3,4,5)$

$\{1,2\} \to \{2,3\} \to \{3,4\} \to \{4,5\} \to \{1,5\}$ (↺ back to $\{1,2\}$)

$\{1,3\} \to \{2,4\} \to \{3,5\} \to \{1,4\} \to \{2,5\}$ (↺ back to $\{1,3\}$)

$|X_g| = 2^2$

Summary

$|X_{()()()()()}| = 2^{10}$ nr perms of this cycle type is 1.

$|X_{(,)(,)()}| = 2^6$ — " — 15

$|X_{(,)()()()}| = 2^7$ — " — 10

$|X_{(,,)()()}| = 2^4$ — " — 20

$|X_{(,,)(,)}| = 2^3$ — " — 20

$|X_{(,,,)()}| = 2^3$ — " — 30

$|X_{(,,,,)}| = 2^2$ — " — 24

$$\frac{1}{|S_5|} \sum_{x \in S_5} |X_g| = \frac{1}{120}\left(1\cdot 2^{10} + 15\cdot 2^6 + 10\cdot 2^7 + 20\cdot 2^4 + 20\cdot 2^3 + 30\cdot 2^3 + 24\cdot 2^2\right)$$

$$= 34$$

Note: We used

$|X_{g_1}| = |X_{g_2}|$

if g_1 and g_2 are

conjugate (see prev HW)

To show

$u = (1,2,3)(4,5)$ is

conjugate with

$x = (a_1, a_2, a_3)(a_4, a_5)$

we see that for

$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}$

we have

$h^{-1} u h = x$

Similarly

$u = (1,2)(3,4)(5)$ is

conjugate with

$x = (a_1, a_2)(a_3, a_4)(a_5)$

we see that for

$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}$ we have $h^{-1} u h = x$

Here $a_i \neq a_j$ for $i \neq j$

and $a_i \in \{1,2,3,4,5\}$

$= [5]$

HW 10.1

Prove that

$(1,2)(3)(4)(5)$ is conjugate

with

$(a_1, a_2)(a_3)(a_4)(a_5)$ for

any choice of $a_i \in \{1,2,3,4,5\}$

and $a_i \neq a_j$ when $i \neq j$.

HW 10.2

Same as HW 10.1

for $(1,2,3)(4)(5)$

HW 10.3

Same for $(1,2,3,4)(5)$

and for $(1,2,3,4,5)$

HW 10.5

Let $u = (1,2,3)(4,5)$ and $x = (1,4,2)(3,5)$

Find explicit h such that

$$h^{-1} u h = x.$$

HW 10.6 Show that the number of unlabeled graphs with 5 vertices is 34.

Do this by listing each graph explicitly

i.e.: