EXERCISES-06

(1) Let $n \in \mathbb{N}$ and k be a fixed positive integer. Recall that
$$\overline{p}(n, k) = \# \text{ of partitions of } n \text{ into at most } k \text{ parts},$$
and
$$p^1(n, k) = \# \text{ of partitions of } n \text{ with each part} \leq k.$$
Derive the generating function $\sum_{n \geq 0} \overline{p}(n, k) q^n$ and show that $\overline{p}(n, k) = p^1(n, k)$.

(2) Give a combinatorial proof of $\overline{p}(n, k) = p^1(n, k)$.
Hint: Define a map explicitly between the sets that respectively enumerate $\overline{p}(n, k)$ and $p^1(n, k)$ and show that the map is bijective.

(3) Let $p_d(n)$ (respectively $p_o(n)$) denote the total number of partitions into distinct parts (respectively odd parts). Give a combinatorial proof of $p_d(n) = p_o(n)$ by constructing a bijection between the sets.