45. Determine if the following sequences are polynomial, C-finite, hypergeometric or holonomic.

    (a) The sequence $(-1)^{\lfloor \log(n+1) \rfloor} + 1$.

    (b) $\sum_{k=-3}^{2n+1} F_{3k+1}^2$ where F_n denotes the Fibonacci sequence.

    (c) The coefficient sequence a_n of $\tan(x) = \sum_{n \geq 0} a_n x^n$.

    (d) The coefficient sequence a_n of $\sqrt{x + 2023} = \sum_{n \geq 0} a_n x^n$.

    (e) The interlacing of the sequences $(0)_{n \geq 0}$ and $(n)_{n \geq 0}$, i.e., the sequence

    $$0, 0, 0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots$$

    It is sufficient to give an argument how one could go about to argue that certain sequences are not in some certain class (without actually doing the rigorous proof).

46. There are $n \geq 2$ students who want to take the final exam of the lecture algorithmic combinatorics. There are three different versions of the exam and the students sit in a circle. Let a_n denote the number of ways to distribute the exams to the students in such a way that two students sitting next to each other do not have the same exam.

    (a) Determine a C-finite recurrence for a_n.

    (b) Determine a closed form expression of a_n.

    (c) Determine the generating function of a_n.

47. Let the Euler operator be defined as $\theta_x = x D_x$. Show that

    $$\theta_x(\theta_x - 1) \cdots (\theta_x - k + 1) f(x) = x^k f^{(k)}(x)$$

    for all formal power series $f(x) = \sum_{n \geq 0} a_n x^n$.

48. Let $(f_n)_{n \geq 0}$ and $(g_n)_{n \geq 0}$ be sequences. Prove that the difference operator satisfies the product rule

    $$\Delta_n(f_n g_n) = (\Delta_n f_n) g_n + f_{n+1}(\Delta g_n), \qquad n \geq 0.$$

    Use this law to deduce the formula for summation by parts,

    $$\sum_{k=0}^{n} f_k(\Delta_k g_k) = f_{n+1} g_{n+1} - f_0 g_0 - \sum_{k=0}^{n} (\Delta_k f_k) g_{k+1}.$$