25. Determine the hypergeometric series representations of

$$\tfrac{1}{x}\log(1+x) = \sum_{n\geq 0} \frac{(-1)^n}{n+1} x^n \qquad \text{and} \qquad \cos(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!} x^{2n}.$$

26. Characterize all sequences that are both C-finite and hypergeometric.

Gauß hypergeometric functions are hypergeometric series of the form

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x\right) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \tag{1}$$

27. Let $f(x)$ be the hypergeometric series from equation (1). Show that $f(x)$ satisfies the differential equation

$$x(1-x)f''(x) + (c - (a+b+1)x)f'(x) - abf(x) = 0.$$

28. Determine the asymptotics of

$$a_n = \binom{2n+1}{n}^2 3^n (n+1),$$

i.e., find a simpler sequence b_n with $a_n \sim b_n (n \to \infty)$. Check on the computer how fast the quotient $\frac{a_n}{b_n}$ actually converges to 1.