21. (Theorem 42) Show that a sequence $(a_n)_{n\geq 0}$ in $\mathbb{K}$ satisfies a C-finite recurrence

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

with $c_i \in \mathbb{K}, c_0 \neq 0$, if and only if

$$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0 x^r}$$

for some polynomial $p(x)$ with degree at most $r-1$.

22. Let $a_n = F_n L_n$, where $(F_n)_{n\geq 0}$ denotes the sequence of Fibonacci numbers and $(L_n)_{n\geq 0}$ the sequence of Lucas numbers. Derive a C-finite recurrence that is satisfied by $(a_n)_{n\geq 0}$.

23. Let $(F_n)_{n\geq 0}$ denote the sequence of Fibonacci numbers. Prove

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

24. Suppose we have a C-finite sequence $(a_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}}$ of order r with companion matrix $C \in \mathbb{R}^{r\times r}$. Then, there is an invertible matrix $S \in \mathbb{C}^{r\times r}$ and a Jordan normal form

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix} \in \mathbb{C}^{r\times r}$$

where each Jordan block is a square matrix of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

such that $C = SJS^{-1}$. The λ_i are the eigenvalues of C, the number of Jordan blocks for an eigenvalue λ_i is its geometric multiplicity (i.e., the dimension of the corresponding eigenspace) and the sum of sizes of the Jordan blocks for an eigenvalue λ_i is its algebraic multiplicity.

  (a) Show that the geometric multiplicity of each eigenvalue is 1.
  (b) Find a formula for J^n.
  (c) Show that every C-finite sequence has a closed form in the sense of Theorem 40.

Bonus: Show that the sequence of Harmonic numbers $(H_n)_{n\geq 0}$ is not C-finite.