13. Two sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ which are non-zero from some index n_0 on are called asymptotically equivalent if $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$. We write $a_n \sim b_n (n \to \infty)$.

    (1) Show that $\sim$ is an equivalence relation.

    (2) Show that for every rational function $r(x) \in \mathbb{K}(x)$ there is a $c \in \mathbb{K}$ and $k \in \mathbb{Z}$ such that $r(n) \sim cn^k$.

    (3) Show that the harmonic numbers H_n cannot be expressed as a rational function, i.e., there is no rational function $r(x) \in \mathbb{K}(x)$ such that $r(n) = H_n$ for all $n \in \mathbb{N}$.

14. Determine a closed form of the exponential generating function $F(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n$ for

    (a) $c_n = n^2$,

    (b) $c_n = \frac{1}{n+1}$.

Recall the definition of *Stirling numbers of the second kind* $S_2(n,k)$ as the number of ways to partition an n-element set into a disjoint union of k nonempty subsets. They satisfy the recurrence relation

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k), \qquad n,k \geq 1,$$

with initial values $S_2(0,0) = 1$ and $S_2(n,0) = 0$ for $n \geq 1$, and $S_2(n,k) = 0$ for $k > n \geq 1$.

15. Show that for $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} S_2(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\mathrm{e}^x - 1\right)^k,$$

and that

$$\sum_{n,k=0}^{\infty} S_2(n,k)\frac{x^n}{n!}y^k = \exp\left(y(\mathrm{e}^x - 1)\right).$$

16. Let the signless Stirling numbers of the first kind $C(n,k)$ denote the number of permutations of $\{1,2,\ldots,n\}$ with exactly k cycles. Derive a recurrence relation for $C(n,k)$. Starting from this recurrence, derive a recurrence relation for the Stirling numbers of the first kind $S_1(n,k) := (-1)^{n-k}C(n,k)$.