

9. Prove Theorem 18: Show that for the formal derivation and integration as defined in the lecture and for all $a(x) \in \mathbb{K}[[x]]$ it holds that

$$(1) D_x \int_x a(x) = a(x)$$

$$(2) \int_x D_x a(x) = a(x) - a(0)$$

$$(3) [x^n]a(x) = \frac{1}{n!} (D_x^n a(x)) \Big|_{x=0}$$

10. Prove Theorem 19 (Multiplicative Inverse):

Let $a(x) \in \mathbb{K}[[x]]$. Then, there exists a series $b(x) \in \mathbb{K}[[x]]$ with $a(x)b(x) = 1$ if and only if $a(0) \neq 0$.

11. Let $(a_n(x))_{n \geq 0}, (b_n(x))_{n \geq 0}$ be convergent sequences of formal power series with respective limits $a(x), b(x) \in \mathbb{K}[[x]]$.

Show that then also $(c_n(x))_{n \geq 0}$ with $c_n(x) = a_n(x) + b_n(x)$ is a convergent sequence of formal power series with limit $a(x) + b(x)$.

12. If asked “Prove that the harmonic numbers are a rational function.”, ChatGPT gives the following proof:

The statement “the harmonic numbers are a rational function” is actually true.

To prove that the harmonic numbers are a rational function, we can use the following fact from calculus: the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

converges for $-1 < x \leq 1$.

Setting $x = 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} = -\ln(0)$$

which diverges, so the harmonic series itself diverges. However, we can still define a rational function that approximates the harmonic series. Specifically, we can define

$$H_n = \sum_{i=1}^n \frac{1}{i} = \frac{1}{n} \sum_{i=1}^n \frac{n}{i}$$

Now, consider the polynomial

$$q(x) = \prod_{i=1}^n (x - i) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where the coefficients a_i are given by elementary symmetric polynomials. In particular, we have

$$a_0 = (-1)^n n!$$

and

$$a_{n-1} = - \sum_{i=1}^n i.$$

Now, define the polynomial

$$p(x) = \sum_{i=0}^{n-1} \frac{a_i}{i!} x^i.$$

We claim that

$$H_n = \frac{p(n)}{q(n)}.$$

To see why, note that

$$\frac{p(n)}{q(n)} = \frac{\sum_{i=0}^{n-1} \frac{a_i}{i!} n^i}{n^n + a_{n-1}n^{n-1} + \cdots + a_1n + a_0}$$

Now, consider the term $\frac{n^k}{k!}$ in the numerator. This term is equal to

$$\frac{n^k}{k!} = \frac{1}{k!} \prod_{i=1}^k n = \frac{1}{k!} \prod_{i=n-k+1}^n i = \frac{1}{k} \binom{n}{k}$$

Therefore, we have

$$\frac{p(n)}{q(n)} = \frac{\sum_{k=1}^n \frac{1}{k} \binom{n}{k}}{\sum_{k=1}^n \frac{1}{k}} = \frac{1}{n} \sum_{k=1}^n \frac{n}{k} = H_n$$

as desired. Therefore, the harmonic numbers are a rational function.

Try to spot all the problems in this “proof”.