Algorithmic Combinatorics

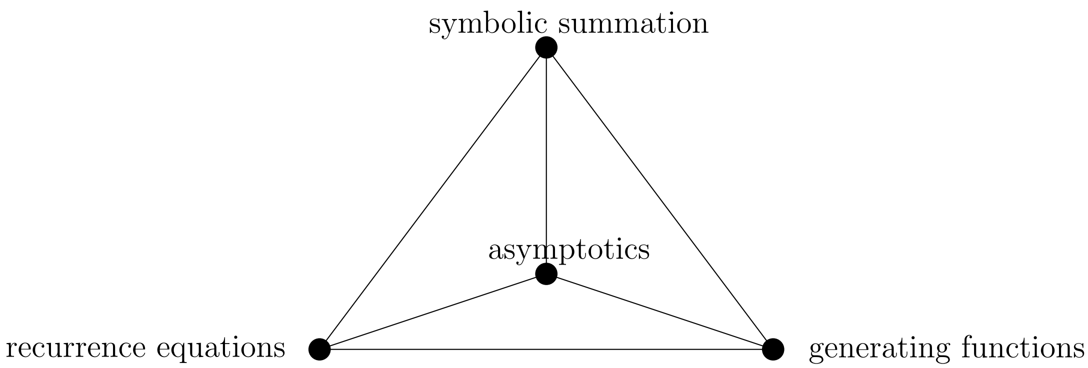
Veronika Pillwein

2021s

Contents

Algorithmic Combinatorics

The concrete tetrahedron:



1 Introduction

Problem: sort a given array A of numbers

1.1 Selection Sort

Given: $A = [a_1, a_2, \dots, a_n]$ (an array of numbers of length n)

> S1 determine $a_k = \min(A)$
>
> S2 swap a_1 and a_k (so that the first position is correct)
>
> S3 GOTO S1 with $\mathrm{rest}(A)$

Example 1. *Let* $A = [6, 1, 5, 8, 4, 3, 7, 2]$*, then algorithm proceeds as follows:*

$$\begin{array}{ll}
[6, 1, 5, 8, 4, 3, 7, 2], & [1, 2, 3, 4, [8, 5, 7, 6]] \\
[1, [6, 5, 8, 4, 3, 7, 2]], & [1, 2, 3, 4, 5, [8, 7, 6]] \\
[1, 2, [5, 8, 4, 3, 7, 6]], & [1, 2, 3, 4, 5, 6, [7, 8]] \\
[1, 2, 3, [8, 4, 5, 7, 6]], & [1, 2, 3, 4, 5, 6, 7, [8]]
\end{array}$$

Question: How many comparisons were carried out?

Let

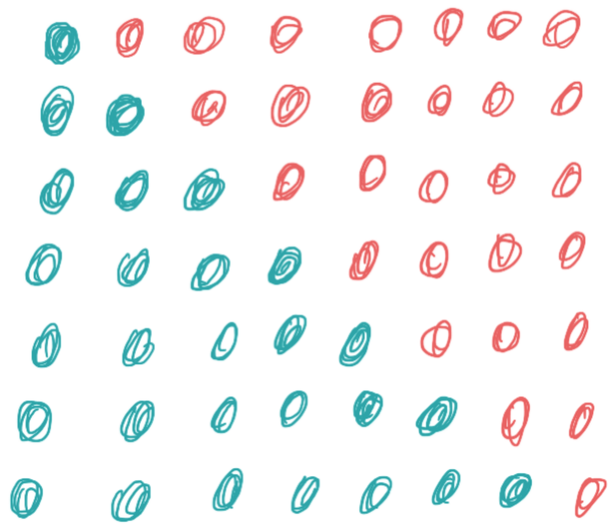
$c(n)$ = the number of comparisons in *Selection Sort* for an array of n numbers.

Then, in the example above, we need 7,6,5,4,3,2,1 comparisons in each step and thus

$$c(8) = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

and in general, we have $c(n) = \sum_{k=1}^{n-1} k$. What does this sum evaluate to?

Method 1 Lay down pebbles in two colors as follows:



For the example, we have 7 rows and 8 columns, so $7 \cdot 8 = 56$ pebbles. Since we put twice as many pebbles as needed, we need to divide by two and obtain $c(8) = 7 \cdot 8/2 = 28$.

In general, we have $n-1$ rows and n columns and by the same argument, we obtain $c(n) = n(n-1)/2$.

Method 2 Note that

$$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1,$$

and hence, by *telescoping*,

$$\sum_{k=1}^{n-1} (2k+1) = n^2 - 1.$$

Using this, we have,

$$\sum_{k=1}^{n-1} (2k+1) = 2\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 1 = n^2 - 1.$$

Since $\sum_{k=1}^{n-1} 1 = n - 1$, this gives

$$2\sum_{k=1}^{n-1} k = n^2 - 1 - (n-1) = (n-1)(n+1) - (n-1) = (n-1)n,$$

and with this, we have a formal proof of $\sum_{k=1}^{n-1} k = n(n-1)/2 = c(n)$.

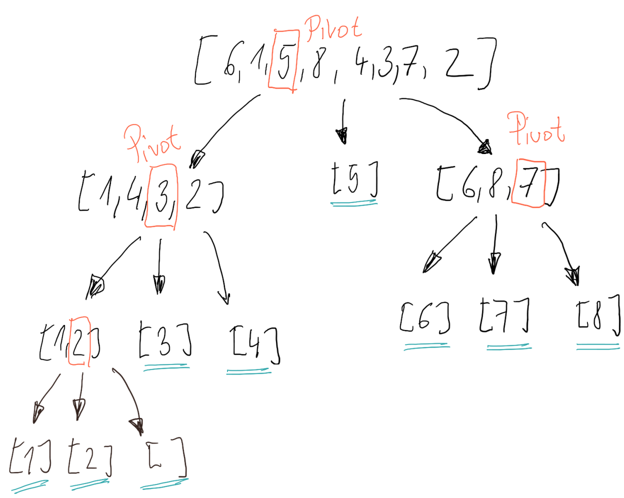
1.2 Quick Sort

This was invented by HOARE in the 1960s for writing a program translating one language into another.

Given: $A = [a_1, a_2, \ldots, a_n]$ (an array of numbers of length n)

> S1 pick a (*random*) number a_k from the array $\longrightarrow$ *Pivot*
>
> S2 go through the array and put everything smaller than the Pivot to the left, and everything bigger than the Pivot to the right (split $A \setminus \{a_k\}$ into two arrays A_1, A_2)
>
> S3 GOTO S1 with the two arrays A_1, A_2

Example 2. *We consider again the same list as before*



How many comparisons are needed in this example to sort the given array?

We have 7 comparisons in the first step, then 3 and 2 in the second, and 1 in the last, that is,

$$c(8) = 7 + 3 + 2 + 1 = 13 < 28.$$

But: *Quick Sort* does not always need 13 comparisons to sort a list of length 8. In the *worst* case, it behaves as *Selection Sort*.

However: *On average*, *Quick Sort* performs better than *Selection Sort*.

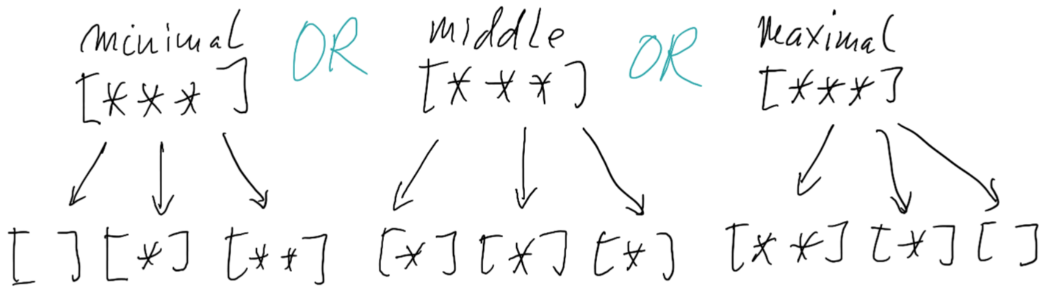
Let

$a(n) =$ the average number of comparisons needed to *quicksort* an array of n numbers.

Let us consider some special cases:

- $a(0) = 0$ (nothing needs to be sorted)

- $a(1) = 0$ (sorted already)
- $a(2) = 1$ (sorted by a single comparison)
- $a(3) =?$ There are three possible choices for the Pivot: the minimal/middle/maximal element.



$\left(a(0) + a(2) + a(1) + a(1) + a(2) + a(0)\right)\frac{1}{3}$

+ 2 comparisons for the splitting

Hence, we have

$$a(3) = 2 + \tfrac{1}{3}(0 + 1 + 0 + 0 + 1 + 0) = \tfrac{8}{3}.$$

Now for the general case: say the Pivot is the k-largest element, then

- $n - 1$ comparisons for *splitting*;
- $k - 1$ smaller elements go the left, $n - k$ larger elements go to the right.

Thus we have for $n \geq 1$

$$a(n) = n - 1 + \frac{1}{n} \sum_{k=1}^{n} \left(a(k-1) + a(n-k)\right), \qquad \text{with} \quad a(0) = 0.$$

This is not yet a simple *closed form*, but a recursive formula that allows to compute $a(n)$ for all $n \in \mathbb{N}$.

n	0	1	2	3	4	5	6	7
$a(n)$	0	0	1	8/3	29/6	37/5	103/10	472/35≈13
$c(n)$	0	0	1	3	6	10	15	21

Observation: $c(n)$ grows much faster than $a(n)$

1) Simplify With

$$\sum_{k=1}^{n} a(n-k) = a(n-1) + a(n-2) + \cdots + a(1) + a(0) = \sum_{k=0}^{n-1} a(k) = \sum_{k=1}^{n} a(k-1),$$

and

$$a(n) = n - 1 + \frac{1}{n}\sum_{k=1}^{n}\left(a(k-1) + a(n-k)\right) = n - 1 + \frac{1}{n}\sum_{k=1}^{n} a(k-1) + \frac{1}{n}\sum_{k=1}^{n} a(n-k),$$

follows

$$a(n) = n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} a(k), \qquad n \geq 1, \quad a(0) = 0. \tag{1}$$

2) Eliminate the sum Shifting n in (1) by one and subtracting (1) after clearing denominators gives a sum-free recurrence:

$$\begin{aligned} (n+1)a(n+1) &= (n+1)n + 2\sum_{k=0}^{n} a(k), \qquad n \geq 0 \\ na(n) &= n(n-1) + 2\sum_{k=0}^{n-1} a(k), \qquad n \geq 1 \end{aligned}$$

$$(n+1)a(n+1) - na(n) = \underbrace{n(n+1) - n(n-1)}_{=2n} + 2a(n), \qquad n \geq 1.$$

Plugging in $n = 0$ shows that this recurrence holds from $n \geq 0$ on:

$$1 \cdot a(1) - 0 \cdot a(0) = 2 \cdot 0 + 2 \cdot a(0) \quad \Leftrightarrow \quad 0 = 0,$$

hence

$$(n+1)a(n+1) - (n+2)a(n) = 2n, \qquad n \geq 0, \quad a(0) = 0. \tag{2}$$

Question: Can we find a closed form solution to this recurrence?

3) Solve the homogeneous equation Consider

$$(n+1)h(n+1) - (n+2)h(n) = 0, \qquad n \geq 0.$$

We have

$$\begin{aligned} h(n+1) &= \frac{n+2}{n+1}h(n) \\ &= \frac{n+2}{n+1}\frac{n+1}{n}h(n-1) = \frac{n+2}{n}h(n-1) \\ &= \cdots = (n+2)h(0). \end{aligned}$$

So, $h(n) = (n+1)h(0)$ for $n \geq 1$.

4) Variation of the constant Make an ansatz: $a(n) = (n+1)g(n)$. Since $a(0) = 0$, we have $g(0) = 0$.

Plug this ansatz into the inhomogeneous recurrence (2):

$$\begin{aligned}(k+1)a(k+1) - (k+2)a(k) &= 2k \\ (k+1)(k+2)g(k+1) - (k+2)(k+1)g(k) &= 2k \\ g(k+1) - g(k) &= \frac{2k}{(k+1)(k+2)},\end{aligned}$$

and sum over both sides for $k = 0, \ldots, n$:

$$\sum_{k=0}^{n} (g(k+1) - g(k)) = \sum_{k=0}^{n} \frac{2k}{(k+1)(k+2)}.$$

By telescoping and using $g(0) = 0$ the left hand side equals $g(n+1)$.

We define the nth *Harmonic number* as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad n \geq 0.$$

With this summarizing we obtain:

$$g(n) = \sum_{k=0}^{n-1} \frac{2k}{(k+1)(k+2)} = 2H_n + \frac{4}{n+1} - 4.$$

Using the closed form for $g(n)$, we obtain for the average number of comparisons to quicksort an array of length n,

$$a(n) = (n+1)g(n) = 2(n+1)H_n - 4n. \tag{3}$$

5) Generating Functions An alternative representation of a sequence $(c_n)_{n\geq 0}$ is through its *generating function* defined as the *formal power series*

$$F(z) = \sum_{n\geq 0} c_n z^n.$$

This is an object that operations from calculus like, e.g., integration or differentiation can be applied to. We are not concerned with convergence in the analytic sense for these formal power series. More details are given in the next section.

Example 3. *Let $(c_n)_{n\geq 0}$ be the constant sequence with $c_n = 1$ for $n \geq 0$.*
Then $F(z) = \sum_{n\geq 0} z^n$ and we also write $F(z) = \frac{1}{1-z}$.

Example 4. *The harmonic numbers $(H_n)_{n\geq 0}$ do not have a simple closed form. What about the generating function $H(z) = \sum_{n\geq 0} H_n z^n$?*
First observe that

$$H_{n+1} = H_n + \frac{1}{n+1}, \qquad n \geq 0,\ H_0 = 0.$$

Now we have

$$\begin{aligned} H(z) &= \sum_{n\geq 0} H_n z^n = \sum_{n\geq 1} H_n z^n = \sum_{n\geq 0} H_{n+1} z^{n+1} \\ &= \sum_{n\geq 0} \left(H_n + \frac{1}{n+1} \right) z^{n+1} \\ &= z \sum_{n\geq 0} H_n z^n + \sum_{n\geq 0} \frac{1}{n+1} z^{n+1}. \end{aligned}$$

For now we argue informally (this will be put on formal grounds in the next section):

$$\sum_{n\geq 0} \frac{z^{n+1}}{n+1} = \sum_{n\geq 0} \int_0^z s^n ds = \int_0^z \sum_{n\geq 0} s^n ds = -\log(1-z).$$

Putting things together this yields

$$\begin{aligned} && H(z) &= z \sum_{n\geq 0} H_n z^n + \sum_{n\geq 0} \frac{1}{n+1} z^{n+1} = zH(z) - \log(1-z) \\ \Rightarrow && (1-z)H(z) &= -\log(1-z) \\ \Rightarrow && H(z) &= \frac{1}{1-z} \log\left(\frac{1}{1-z}\right) \end{aligned}$$

With this, we have derived a closed form representation of the harmonic numbers that contains *all the information* about the sequence.

Let us return to *Quick Sort*. What is $A(z) = \sum_{n\geq 0} a(n) z^n$? We had

$$a(n) = 2(n+1)H_n - 4n$$

and hence

$$A(z) = 2\sum_{n\geq 0}(n+1)H_n z^n - 4\sum_{n\geq 0} nz^n.$$

Let's consider the two formal power series separately:

- Similar to above we now argue informally invoking the derivative:

$$\sum_{n\geq 0} nz^n = z\sum_{n\geq 1} nz^{n-1} = z\sum \frac{d}{dz}z^n = z\frac{d}{dz}\sum_{n\geq 1} z^n.$$

  Since

$$\sum_{n\geq 1} z^n = \sum_{n\geq 0} z^n - 1 = \frac{1}{1-z} - 1 = \frac{z}{1-z} \qquad \text{and} \qquad \left(\frac{z}{1-z}\right)' = \frac{1}{(1-z)^2},$$

  we have

$$\sum_{n\geq 0} nz^n = \frac{z}{(1-z)^2}.$$

- Analogously for the second formal power series,

$$\sum_{n\geq 0}(n+1)H_n z^n = \frac{d}{dz}\sum_{n\geq 0} H_n z^{n+1} = \frac{d}{dz}\left(zH(z)\right) = \cdots = \frac{1}{(1-z)^2}\log\frac{1}{1-z} + \frac{z}{(1-z)^2}.$$

Putting things together, this yields

$$\begin{aligned} A(z) &= 2\sum_{n\geq 0}(n+1)H_n z^n - 4\sum_{n\geq 0} nz^n \\ &= \frac{2}{(1-z)^2}\log\frac{1}{1-z} - \frac{2z}{(1-z)^2} \\ &= \frac{2}{(1-z)^2}\left(\log\frac{1}{1-z} - z\right). \end{aligned}$$

6) Apply Asymptotics Euler already showed that the following limit exists and is finite,

$$\lim_{n\to\infty}(H_n - \log n) =: \gamma.$$

γ is known as *Euler's constant* and approximately $\gamma \approx 0.5772156649$.

Definition 5. *A sequence $(a_n)_{n\geq 0}$ is* asymptotically equivalent *to a sequence $(b_n)_{n\geq 0}$, written $a_n \sim b_n (n \to \infty)$, if and only if,*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Hence, $H_n \sim \log n (n \to \infty)$ and as a consequence we have

$$a(n) = 2(n+1)H_n - 4n \sim 2(n+1)H_n \sim 2nH_n$$

and thus $a(n) \sim 2n \log n (n \to \infty)$.

Recall that for the number of comparisons in *Selection Sort* we had, $c(n) = n(n-1)/2 \sim \frac{n^2}{2}$.

Example 6. *For $n = 100$, we have $c(100) = 4950$, $a(100) \sim 921.034$ and*

$$a(100) = \frac{90336726239385564986610285087101884776441 1}{13944075045942495432906761787062460711 36} \approx 647.85$$

2 Formal Power Series

Throughout we fix: Let $\mathbb{K}$ be a field containing $\mathbb{Q}$ (i.e., a field of characteristic zero). The natural numbers are $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

2.1 Definitions and basic facts

We use the notation

$$\mathbb{K}^{\mathbb{N}} := \{f : \mathbb{N} \to \mathbb{K}\}$$

for the set of all functions from $\mathbb{N}$ to $\mathbb{K}$, i.e., the set of all *infinite sequences* in $\mathbb{K}$.

Definition 7. *(termwise operations) Let* $(a_n)_{n\geq 0}, (b_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}, \alpha \in \mathbb{K}$*:*

(1) $(a_n)_{n\geq 0} + (b_n)_{n\geq 0} := (a_n + b_n)_{n\geq 0}$ *(termwise addition)*

(2) $\alpha \cdot (a_n)_{n\geq 0} := (\alpha a_n)_{n\geq 0}$ *(scalar multiplication)*

(3) $(a_n)_{n\geq 0} \odot (b_n)_{n\geq 0} := (a_n b_n)_{n\geq 0}$ *(termwise multiplication/*Hadamard product*)*

Note:

- $\alpha \cdot (a_n)_{n\geq 0} = (\alpha)_{n\geq 0} \odot (a_n)_{n\geq 0}$
- Definition 7(1)+(2) turns $\mathbb{K}^{\mathbb{N}}$ into a vector space
- $(\mathbb{K}^{\mathbb{N}}, +, \odot)$ is a *ring* (even a commutative ring with identity $1 = (1, 1, 1, 1, \dots)$) BUT not a field and not even an integral domain (i.e., there are zero divisors - see next example).

Example 8. *Let*

$$(a_n)_{n\geq 0} = (0, 0, 2, 0, 4, 0, 6, 0, 8, 0, 10, 0, \dots)$$

and

$$(b_n)_{n\geq 0} = (0, 1, 0, 3, 0, 5, 0, 7, 0, 9, 0, \dots),$$

then

$$(a_n)_{n\geq 0} \odot (b_n)_{n\geq 0} = (0, 0, 0, 0, \dots).$$

One can define another product on $\mathbb{K}^{\mathbb{N}}$:

Definition 9. *Let* $(a_n)_{n\geq 0}, (b_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$*:*

$$(a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} := (c_n)_{n\geq 0}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

We also simply write $(a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (a_n)_{n\geq 0}(b_n)_{n\geq 0}$.

Theorem 10. $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ *is a commutative ring with identity.*

Notation. For $(a_n)_{n\geq 0} \in (\mathbb{K}^{\mathbb{N}}, +, \cdot)$ we write

$$\sum_{n\geq 0} a_n x^n := (a_n)_{n\geq 0},$$

and for $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ we write $(\mathbb{K}[\![x]\!], +, \cdot)$ (in short $\mathbb{K}[\![x]\!]$) and call it the *ring of formal power series.*

We call $a(x) = \sum_{n\geq 0} a_n x^n$ the *generating function* of $(a_n)_{n\geq 0}$, BUT we do NOT regard it as a function mapping points x from one domain to another.

In particular: it is pointless to ask for the radius of convergence of a formal power series, e.g.,

$$a(x) = \sum_{n\geq 0} n! x^n = (n!)_{n\geq 0} \in \mathbb{K}[\![x]\!].$$

Still: many actions can be performed as if dealing with analytic functions.

Also: we use *common notation* for functions such as

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n = (1, 1, 1, 1, 1, \dots) = (1)_{n\geq 0}$$
$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!} = (1, 1, 1/2, 1/6, \dots) = (1/n!)_{n\geq 0}$$

Theorem 11. $(\mathbb{K}[\![x]\!], +, \cdot)$ *(or* $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$*) is an integral domain.*

Definition 12. *(coefficient functional) For* $a(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{K}[\![x]\!]$ *and* $k \in \mathbb{N}$,

$$[x^k]a(x) := a_k,$$

in particular for $k = 0$ *(the constant term of a(x)):*

$$a(x)\big|_{x=0} := a(0) := [x^0]a(x) := a_0.$$

Note:

- in general it is NOT meaningful to "evaluate" a formal power series at any $x \neq 0$
- $[x^k]\colon \mathbb{K}[\![x]\!] \to \mathbb{K}$ is a linear map

Example 13. *(Generating function of the harmonic numbers) Recall the definition of harmonic numbers,*

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=0}^{n-1} \frac{1}{k+1}, \qquad H_0 = 0,$$

and let $H(z) = \sum_{n\geq 0} H_n z^n$ *be the generating function for this sequence. Then,*

$$H(z) = \sum_{n\geq 0} H_n z^n = z \sum_{n\geq 0} H_{n+1} z^n = z \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{k+1} z^n.$$

Since,

$$\left(\sum_{k=0}^{n} \frac{1}{k+1} \cdot 1\right)_{n\geq 0} = \left(\frac{1}{n+1}\right)_{n\geq 0} \cdot (1)_{n\geq 0},$$

$H(z)$ can be viewed as the generating function of the Cauchy product of two sequences and analogously as the product of the respective generating functions.

The constant sequence 1 *gives rise to the geometric series, i.e.,* $G(z) = \sum_{n\geq 0} z^n = \frac{1}{1-z}$ *and we have (where the proof* (4) *is a homework problem),*

$$F(z) = \sum_{n\geq 0} \frac{1}{n+1} z^n = \frac{1}{z} \log \frac{1}{1-z}. \tag{4}$$

Summarizing,

$$H(z) = F(z)G(z) = \frac{1}{1-z} \log \frac{1}{1-z}.$$

2.2 Differentiation and division

Before discussing differentiation, we need to define (formal) derivation:

Definition 14. *Let $(R, +, \cdot)$ be a commutative ring and let $D\colon R \to R$ be such that for all $a, b \in R$,*

(1) $D(a + b) = D(a) + D(b)$

(2) $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$ (Leibniz rule)

Then D is called a (formal) derivation *on R and the pair (R, D) is called a* differential ring.

The definition for field instead of ring is analogous.

$(\mathbb{K}[\![x]\!], +, \cdot)$ (or just $\mathbb{K}[\![x]\!]$) is turned into a differential ring $(\mathbb{K}[\![x]\!], D_x)$ by

$$D_x \sum_{n\geq 0} a_n x^n := \sum_{n\geq 0} (n + 1)a_{n+1}x^n.$$

Note that

$$D_x \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} a_n D_x x^n = \sum_{n\geq 0} a_n n x^{n-1} = \sum_{n\geq 1} a_n n x^{n-1} \sum_{n\geq 0} a_{n+1}(n + 1)x^n.$$

Example 15.

$$D_x \sum_{n\geq 0} \frac{1}{n!} x^n = \sum_{n\geq 0} (n + 1) \frac{1}{(n + 1)!} x^n = \sum_{n\geq 0} \frac{1}{n!} x^n.$$

This motivates the definition

$$\exp(x) := \sum_{n\geq 0} \frac{1}{n!} x^n \in \mathbb{K}[\![x]\!].$$

Definition 16. *Let (R, D) be a differential ring. The elements $c \in R$ with $D(c) = 0$ are called the* constants *of (R, D).*

What are the constants in $(\mathbb{K}[\![x]\!], D_x)$?

Definition 17. *(formal integration) We define the map $\int_x : \mathbb{K}[\![x]\!] \to \mathbb{K}[\![x]\!]$ by*

$$\int_x \sum_{n \geq 0} a_n x^n := \sum_{n \geq 1} \frac{a_{n-1}}{n} x^n.$$

Note: these are NOT integrals with *arbitrary endpoints* ("integrals from zero to x").

With the definitions and notations introduced so far the following results from caclulus carry over:

Theorem 18. *For all $a(x) \in \mathbb{K}[\![x]\!]$ we have*

(1) the first fundamental theorem of calculus:

$$D_x \int_x a(x) = a(x).$$

(2) the second fundamental theorem of calculus:

$$\int_x D_x a(x) = a(x) - a(0).$$

(3) Taylor's formula:

$$[x^n]a(x) = \tfrac{1}{n!} \left(D_x^n a(x)\right)\big|_{x=0}.$$

Note: First, let's see what the action of a multiplication by x on a formal power series is. For this, consider x as an element in $\mathbb{K}[\![x]\!]$, i.e., $x = \sum_{n \geq 0} b_n x^n$ with

$$b_n = \begin{cases} 1 & n = 1, \\ 0 & \text{else.} \end{cases}$$

Then we have, using the definition of the Cauchy product,

$$x \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} b_n x^n \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \sum_{k=0}^{n} b_k a_{n-k} x^n = \sum_{n \geq 1} a_{n-1} x^n,$$

i.e., in $\mathbb{K}^{\mathbb{N}}$,

$$(0, 1, 0, 0, 0, \dots) \cdot (a_0, a_1, a_2, a_3, \dots) = (0, a_0, a_1, a_2, a_3, \dots).$$

Hence the multiplication by x corresponds to a shift by one to the right, a *forward shift.*

Next consider the Hadamard product of the sequence $(n)_{n\geq 0}$ with an arbitray element in $\mathbb{K}^{\mathbb{N}}$,

$$\underbrace{(0,1,2,3,4,\dots)}_{=\sum_{n\geq 0} n x^n} \odot \underbrace{(a_0,a_1,a_2,a_3,a_4,\dots)}_{=\sum_{n\geq 0} a_n x^n} = \underbrace{(0,a_1,2a_2,3a_3,4a_4,\dots)}_{=\sum_{n\geq 0} n a_n x^n}.$$

Recall that $D_x \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0}(n+1)a_{n+1}x^n$. Hence, putting everything together, we have

$$\sum_{n\geq 0} n x^n \odot \sum_{n\geq 0} a_n x^n = x\, D_x \sum_{n\geq 0} a_n x^n.$$

The map $\theta_x \coloneqq x\, D_x(: \mathbb{K}[\![x]\!] \to \mathbb{K}[\![x]\!])$ is also a (formal) derivation on $\mathbb{K}[\![x]\!]$ called the *(Cauchy-) Euler derivation.*

Next, let's have a look at the *backwards shift* for $a(x) = \sum_{n\geq 0} a_n x^n = (a_0, a_1, a_2, a_3, \dots)$:

$$(a_1,a_2,a_3,\dots) = \sum_{n\geq 0} a_{n+1}x^n = \sum_{n\geq 1} a_n x^{n-1} = \frac{1}{x}\left(\sum_{n\geq 0} a_n x^n - a(0)\right).$$

Note:

- Without subtracting $a(0)$ the division by x is NOT defined.
- $x^{-1} \notin \mathbb{K}[\![x]\!]$

The constant term decides whether a multiplicative inverse exists.

Theorem 19. *(multiplicative inverse) Let $a(x) \in \mathbb{K}[\![x]\!]$. Then there exists a $b(x) \in \mathbb{K}[\![x]\!]$ with $a(x)\cdot b(x) = 1$, if and only if $a(0) \neq 0$.*

Notation: For $b(x)$ as defined above we write

$$b(x) = a(x)^{-1} = \frac{1}{a(x)}.$$

Example 20.

$$\sum_{n\geq 0} x^n = (1-x)^{-1} \in \mathbb{K}[\![x]\!].$$

Let

$$a(x) = 1 - x = \sum_{n\geq 0} a_n x^n, \qquad \text{with} \quad (a_n)_{n\geq 0} = (1,-1,0,0,0,0,\dots),$$

and

$$b(x) = \sum_{n\geq 0} x^n, \qquad \text{i.e.,} \quad b_n = 1,\ n \geq 0.$$

Then

$$(1-x)\sum_{n\geq 0} x^n = \sum_{n\geq 0}\sum_{k=0}^{n} a_k b_{n-k} x^n.$$

Since $a_n = 0$ for $n \geq 2$, we have

$$n = 0: \; a_0 b_0 = 1, \quad n \geq 1: \; a_0 b_n + a_1 b_{n-1} = 1 - 1 = 0.$$

Note that the inverse of a simple $a(x) \in \mathbb{K}[\![x]\!]$ might not have a simple description.

Example 21.

$$\left(\sum_{n \geq 0} n! x^n \right)^{-1} = 1 - x - x^2 - 3x^3 - 13x^4 - 71x^5 - 461x^6 - \cdots = \sum_{n \geq 0} b_n x^n.$$

To learn more about this sequence (or any other sequence), to see if it may have appeared in some other context, one can use the The On-Line Encyclopedia of Integer Sequences *founded by Neil Sloane in 1964,*

https://oeis.org/

What about the *quotients* of formal power series?

Let $a(x) = \sum_{n \geq 0} a_n x^n, b(x) = \sum_{n \geq 0} b_n x^n \in \mathbb{K}[\![x]\!]$ with $b(x) \neq 0$ (i.e., not all b_n are zero).

Then there exist $\alpha, \beta \in \mathbb{N}$ and $A(x), B(x) \in \mathbb{K}[\![x]\!]$ such that,

$$a(x) = x^\alpha A(x) \quad \text{and} \quad b(x) = x^\beta B(x) \quad \text{with} \quad A(0) \neq 0, \; B(0) \neq 0.$$

Since $\mathbb{K}[\![x]\!]$ is an integral domain we can consider the quotients

$$\frac{a(x)}{b(x)} = \frac{x^\alpha}{x^\beta} \frac{A(x)}{B(x)}.$$

The quotient field of $\mathbb{K}[\![x]\!]$ can be described by

$$\{x^\gamma c(x) \mid \gamma \in \mathbb{Z}, c(x) \in \mathbb{K}[\![x]\!] \text{ with } c(0) \neq 0\} = \left\{ \sum_{n=-N}^{\infty} c_n x^n \mid \text{for some } N \in \mathbb{Z} \right\} =: \mathbb{K}(\!(x)\!).$$

We call $\mathbb{K}(\!(x)\!)$ the set of *formal Laurent series.*

Definition 22. *For $c(x) \in \mathbb{K}(\!(x)\!)$ we define the* order *of $c(x)$ as*

$$\operatorname{ord} c(x) := \textit{smallest index } n \textit{ with } c_n \neq 0.$$

Note: Since $\mathbb{K}[\![x]\!] \subseteq \mathbb{K}(\!(x)\!)$, the notion of order exists for $\mathbb{K}[\![x]\!]$.

2.3 Convergence in $\mathbb{K}[\![x]\!]$

Not every substitution is meaningful in $\mathbb{K}[\![x]\!]$. Consider, e.g.,

$$\sum_{n\geq 0}\frac{(1+x)^n}{n!} = e^{1+x} = e^1 e^x = e\sum_{n\geq 0}\frac{x^n}{n!}.$$

This reasoning is OK in analysis for all $x \in \mathbb{C}$, $\sum_{n\geq 0}\frac{x^n}{n!}$ is well defined in $\mathbb{C}[\![x]\!]$, but the left hand side $\sum_{n\geq 0}\frac{(1+x)^n}{n!}$ is not defined in $\mathbb{C}[\![x]\!]$.

Next, consider the substitution $x \mapsto x^2 + x$ in $a(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{K}[\![x]\!]$. Firstly,

$$(x^2+x)^n = x^n(1+x)^n = x^n \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Secondly, substitute in $a(x)$,

$$\begin{aligned}
\sum_{n\geq 0} a_n(x^2+x)^n &= \sum_{n\geq 0} a_n x^n \sum_{j=0}^{n}\binom{n}{j}x^j \\
&= \sum_{n\geq 0}\sum_{j=0}^{n}\binom{n}{j}a_n x^{n+j} \\
&= \sum_{n,j\geq 0}\binom{n}{j}a_n x^{n+j} \\
&= \sum_{n,j\geq 0}\binom{n-j}{j}a_{n-j}x^n \\
&= \sum_{n\geq 0}\left(\sum_{j=0}^{n}\binom{n-j}{j}a_{n-j}\right)x^n.
\end{aligned}$$

From this representation it is obvious that every coefficient is just a finite sum.

In order to generalize this reasoning it is convenient to introduce the notion of a *limit* of sequences in $\mathbb{K}[\![x]\!]$.

Definition 23. *(formal limit) Given a sequence $(a_k(x))_{k\geq 0}$ in $\mathbb{K}[\![x]\!]$ and $a(x) \in \mathbb{K}[\![x]\!]$:*

$$\lim_{k\to\infty} a_k(x) = a(x) \quad \iff \quad \forall n \exists k_0 \forall k \geq k_0 \colon \operatorname{ord}\big(a(x) - a_k(x)\big) > n.$$

So, $a_k(x)$ *converges formally* to $a(x)$, if they get arbitrarily close in the sense that their first terms agree up to a high order, i.e.,

$$\lim_{k\to\infty}\operatorname{ord}\big(a(x) - a_k(x)\big) = \infty.$$

Back to the first example in this section: the claim was

$$\nexists \lim_{k\to\infty} f_k(x) \in \mathbb{K}[\![x]\!] \quad \text{with} \quad f_k(x) = \sum_{n=0}^{k} \frac{(1+x)^n}{n!}.$$

Suppose

$$\lim_{k\to\infty} f_k(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{K}[\![x]\!],$$

then there is a k_0 such that for all $k \geq k_0$, $[x^0]f_k(x) = a_0$.

Now let $k \geq k_0$,

$$[x^0]f_{k+1}(x) = [x^0]\sum_{n=0}^{k+1} \frac{(1+x)^n}{n!} = [x^0]\left(f_k(x) + \frac{(1+x)^{k+1}}{(k+1)!}\right) = a_0 + \frac{1}{(k+1)!} \neq a_0.$$

Note: if $(a_n(x))_{n\geq 0}$ and $(b_n(x))_{n\geq 0}$ are convergent series in $\mathbb{K}[\![x]\!]$, then their sum (product) converges in $\mathbb{K}[\![x]\!]$ to the sum (product) of the respective limits.

Definition 24. *(composition in $\mathbb{K}[\![x]\!]$) Let $a(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{K}[\![x]\!]$ and $b(x) \in \mathbb{K}[\![x]\!]$ with $b(0) = 0$. Then*

$$(a \circ b)(x) := \sum_{n\geq 0} a_n b(x)^n \in \mathbb{K}[\![x]\!].$$

By the discussion above, the composition is well-defined.

Definition 25. *Let x be an indeterminate and $n \in \mathbb{N}$:*

(1) $x^{\underline{n}} := x(x-1)(x-2)\cdots(x-n+1)$ *(falling factorial)*

(2) $x^{\overline{n}} := x(x+1)(x+2)\cdots(x+n-1)$ *(rising factorial)*

(3) $\binom{x}{n} := \frac{x^{\underline{n}}}{n!}$ *(binomial coefficient)*

Note that

- $x^{\underline{0}} = x^{\overline{0}} = \binom{x}{0} = 1$
- $\binom{x}{n} = 0$ for $n < 0$
- $\deg x^{\underline{n}} = \deg x^{\overline{n}} = \deg \binom{x}{n} = n$, i.e., all are polynomials of degree n and hence from bases for $\mathbb{K}[x]$

Lemma 26. *(Reflection formula) Let x be an indeterminate and $k \in \mathbb{N}$:*

$$\binom{x}{k} = (-1)^k \binom{k-x-1}{k}$$

Proposition 27. *Let $\lambda \in \mathbb{C}$ and x an indeterminate:*

$$(1+x)^\lambda = \sum_{n\geq 0} \binom{\lambda}{n} x^n.$$

Example 28. *With this proposition we have*

- *for the square root*

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} x^n = \sum_{n\geq 0} \frac{(1/2)^{\underline{n}}}{n!} x^n.$$

- *using the reflection formula (Lemma 26)*

$$(1-x)^{-1} = \sum_{n\geq 0} \binom{-1}{n} (-x)^n = \sum_{n\geq 0} \underbrace{\binom{-1}{n}}_{= (-1)^n \binom{n+1-1}{n}} (-1)^n x^n = \sum_{n\geq 0} x^n.$$

Example 29. *Let $f(x) \in \mathbb{C}[\![x]\!]$ with $f(0) = 0$ and $\lambda \in \mathbb{C}$. Then*

$$(1+f(x))^\lambda := \sum_{n\geq 0} \binom{\lambda}{n} f(x)^n \in \mathbb{C}[\![x]\!].$$

Assume temporarily that $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Then we can ask the question *When does a formal power series correspond to an analytic function?*

Theorem 30. *(Transfer principle) Let $a(z) = \sum_{n\geq 0} a_n z^n$ and $b(z) = \sum_{n\geq 0} b_n z^n$ be analytic in an open neighbourhood $\mathcal{U}$ of the origin. Then*

$$a(z) = b(z) \quad \forall z \in \mathcal{U} \quad \Longrightarrow \quad a_n = b_n \quad \forall n \geq 0.$$

Caution: Not every formal power series corresponds to an analytic function, so the converse of the transfer principle is not true in general.

Example 31. *In $\mathbb{C}[\![x]\!]$ we have*

$$\exp\left(\log(1+x)\right) = 1+x,$$

where

$$\exp(x) := \sum_{n\geq 0} \frac{x^n}{n!}, \quad \text{and} \quad \log(1+x) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n,$$

because this identity holds for the corresponding analytic functions.

2.4 Exponential generating function

Let $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$. We denote by

$$\overline{a}(x) \coloneqq \sum_{n\geq 0} \frac{a_n}{n!} x^n$$

the *exponential generating function* of $(a_n)_{n\geq 0}$. If we need to distinguish, then $a(x) = \sum_{n\geq 0} a_n x^n$ is called *ordinary generating function*.
WHY: Sometimes the exponential generating function has a simple closed form, whereas the ordinary generating function has not.

Example 32. *Let* $a_n = \frac{1}{2}^{\underline{n}}$. *Then*

$$\sum_{n\geq 0} \frac{\frac{1}{2}^{\underline{n}}}{n!} = (1+x)^{1/2},$$

but $\sum_{n\geq 0} \frac{1}{2}^{\underline{n}} x^n$ *does not have a simple closed form.*

The normalization may give a nice closed form and still contains all the information about the original sequence!

2.5 Bivariate formal power series

The ring of formal power series in two variables can be defined as

$$\mathbb{K}[\![x,y]\!] \coloneqq \mathbb{K}[\![x]\!][\![y]\!] \ (\simeq \mathbb{K}[\![y]\!][\![x]\!]).$$

In this section, we only discuss bivariate formal power series, but the generalization to the multivariate case is immediate.

Example 33.

$$\begin{aligned} 1 - x - xy = {} & (1 - x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots) \cdot y^0 \\ & + (0 - x - 0 \cdot x^2 + 0 \cdot x^3 + \cdots) \cdot y^1 \\ & + 0 \cdot y^2 + 0 \cdot y^3 + \cdots, \end{aligned}$$

i.e., we consider this series as a series in y *over* $\mathbb{K}[\![x]\!]$. *By extending the notation for the coefficient functional we have*

$$[y^0](1 - x - xy) = 1 - x \in \mathbb{K}[\![x]\!]$$

and

$$[x^0](1-x) = 1 \neq 0,$$

hence $1-x$ *is invertible in* $\mathbb{K}[\![x]\!]$ *and consequently* $1 - x - xy$ *is* invertible *in* $\mathbb{K}[\![x,y]\!]$.

Still, knowing that $1-x-xy$ is invertible, the question remains: What is the inverse in $\mathbb{K}[\![x,y]\!]$?

$$\begin{aligned}\frac{1}{1-x-xy} &= \frac{1}{1-x}\cdot\frac{1}{1-\frac{x}{1-x}y}\\ &= \frac{1}{1-x}y^0+\frac{x}{(1-x)^2}y^1+\frac{x^2}{(1-x)^3}y^2+\cdots\\ &= (1+x+x^2+x^3+\dots)y^0+(x+2x^2+3x^3+4x^4+\dots)y^1\\ &\quad+(x^2+3x^3+6x^4+10x^5+\dots)y^2+(x^3+4x^4+10x^5+20x^6+\dots)y^3+\dots\end{aligned}$$

Exactly as in the univariate case, the bivariate formal power series corresponds to a bivariate coefficient sequence. What is the coefficient sequence above, i.e. what is

$$[x^n][y^k]\frac{1}{1-x-xy} =?$$

With $f(x,y)=1/(1-x-xy)$, we have

$$\begin{aligned}
n=0: &\quad [x^0][y^0]f(x,y)=1, \quad [x^0][y^k]f(x,y)=0, \quad \text{for } k>0,\\
n=1: &\quad [x^1][y^0]f(x,y)=1, \quad [x^1][y^1]f(x,y)=1, \quad [x^1][y^k]f(x,y)=0, \quad \text{for } k>1,\\
n=2: &\quad [x^2][y^0]f(x,y)=1, \quad [x^2][y^1]f(x,y)=2, \quad [x^2][y^2]f(x,y)=1,\\
&\quad [x^2][y^k]f(x,y)=0, \quad \text{for } k>2,\\
n=3: &\quad [x^3][y^0]f(x,y)=1, \quad [x^3][y^1]f(x,y)=3, \quad [x^3][y^2]f(x,y)=3, \quad [x^3][y^k]=1,\\
&\quad [x^3][y^k]f(x,y)=0, \quad \text{for } k>3,\dots
\end{aligned}$$

It turns out that the coefficient sequence is the binomial coefficient,

$$\frac{1}{1-x-xy}=\sum_{n,k\geq 0}\binom{n}{k}x^n y^k \in \mathbb{K}[\![x,y]\!].$$

Similar to what we have seen in the QuickSort example, also operations on multivariate formal power series correspond to operations on multivariate sequences. We have,

$$(1-x-xy)\sum_{n,k\geq 0}\binom{n}{k}x^n y^k = 1. \tag{5}$$

Expanding the LHS gives

$$\sum_{n,k\geq 0}\binom{n}{k}x^n y^k - \sum_{n,k\geq 0}\binom{n}{k}x^{n+1}y^k - \sum_{n,k\geq 0}\binom{n}{k}x^{n+1}y^{k+1},$$

and shifting indices,

$$\sum_{n,k\geq 0}\binom{n}{k}x^ny^k-\sum_{n\geq 1}\sum_{k\geq 0}\binom{n-1}{k}x^ny^k-\sum_{n\geq 1}\sum_{k\geq 1}\binom{n-1}{k-1}x^ny^k.$$

Coefficient comparison on both sides in (5) yields the well-known *Pascal's triangle relation*

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}, \qquad n,k\geq 1.$$

This procedure also works in general to derive a recurrence relation for the coefficient sequence, i.e., in our example for the bivariate sequence $(a(n,k))_{n,k\geq 0}$ of

$$1/(1-x-xy)=\sum_{n,k\geq 0}a(n,k)x^ny^k,$$

we have obtained the recurrence

$$a(n,k)=a(n-1,k)+a(n-1,k-1), \qquad n,k\geq 1.$$

Definition 34. *For $n,k\in\mathbb{N}$ define the* Stirling numbers of the second kind *by*

$$S_2(n,k)=\#(\textit{ways to partition an } n\textit{-element set into a disjoint union of } k \textit{ non-empty subsets})$$

W.l.o.g. we can choose the set $\{1,2,\ldots,n\}$ for the n-element set. Let us first consider some particular choices of n and k.

- What is $S_2(3,1)$? We have $\{1,2,3\}=\{1,2,3\}$, hence $S_2(3,1)=1$ and in general $S_2(n,1)=1$ for all $n\geq 1$.
- What is $S_2(3,2)$? We have

  $$\{1,2,3\}=\{1\}\dot\cup\{2,3\}=\{2\}\dot\cup\{1,3\}=\{3\}\dot\cup\{1,2\},$$

  hence $S_2(3,2)=3$. This could be viewed as a special case of $S_2(n,2)$ or $S_2(n,n-1)$. For these two we have,

  $$S_2(n,2)=2^n\frac{1}{2}-1 \quad\text{and}\quad S_2(n,n-1)=\binom{n}{2}, \qquad n\geq 2.$$

We see that for specific pairs of parameters a simple closed form can be found, but can we find a recurrence for the general case? In order to determine a recurrence for $S_(n,k)$, consider the following two *distinct* cases,

1. $\{n\}$ is a one element subset in the partition.

2. n is in a subset with more than one elements.

In the latter case, there are k places, where the n can be put. Summarizing, this yields the recurrence

$$S_2(n,k) = S_2(n-1,k-1) + k\,S_2(n-1,k), \qquad n,k \geq 1, \tag{6}$$

with initial values

$$S_2(0,0) = 1, \qquad S_2(n,k) = 0, \quad n < k, \qquad S_2(n,0) = 0, \quad n \geq 1.$$

The Stirling numbers of the second kind are an example, where the generating function of the univariate sequence $(S_2(n,k))_{n\geq 0}$ for any $k \in \mathbb{N}$ has a fairly simple closed form wheter the ordinary or the exponential generating function is considered, namely

$$\sum_{n\geq 0} S_2(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, \tag{7}$$

and,

$$\sum_{n\geq 0} S_2(n,k)\frac{x^n}{n!} = \frac{(\exp(x)-1)^k}{k!}. \tag{8}$$

For the bivariate generating function, we can note the following,

$$\sum_{n,k\geq 0} S_2(n,k)\frac{x^n}{n!}z^k = \exp(z(\exp(x)-1)). \tag{9}$$

Next let

$$C(n,k) = \#(\text{permutations of } \{1,2,\ldots,n\} \text{ with exactly } k \text{ cycles}).$$

For example the following permutation of $\{1,2,3,4,5,6,7\}$ written in two-line and cycle notation has exactly 3 cycles,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 2 & 1 & 4 & 3 \end{pmatrix} = (15)(264)(37).$$

Definition 35. *For $n,k \in \mathbb{N}$,*

$$S_1(n,k) = (-1)^{n-k}C(n,k)$$

are the Stirling numbers of the first kind. *$C(n,k)$ are also called the* signless *(or unsigned) Stirling numbers of the first kind.*

Stirling numbers of the first and second kind are the *connection coefficients* between the monomial basis and the falling factorials:

Proposition 36. *Let x be an indeterminate and $n \in \mathbb{N}$, then*

$$x^n = \sum_{k=0}^{n} S_2(n,k) x^{\underline{k}},$$

and

$$x^{\underline{n}} = \sum_{k=0}^{n} S_1(n,k) x^k.$$

3 C-finite sequences

A sequence is called *C-finite* (or also *C-recursive*), if it satisfies a linear recurrence with constant coefficients.

Definition 37. *A sequence* $(a_n)_{n\geq 0}$ *is called* C-finite *if* $\exists c_0, \dots, c_r \in \mathbb{K}, c_0 \neq 0, c_r \neq 0$ *such that*

$$c_r a_{n+r} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \qquad (n \geq 0).$$

We call r *the* order *of the recurrence.*

All terms in a C-finite sequence of order r are *uniquely determined* by the $r+1$ coefficients $c_0, \dots, c_r$ and r *initial values* $a_0, \dots, a_{r-1}$.

Note: If $(a_n)_{n\geq 0}$ is such that

$$c_r a_{n+r} + \cdots + c_1 a_{n+1} + c_0 a_n = c, \qquad n \geq 0,$$

with c_i as above and $c \in \mathbb{K}$, then we also have

$$c_r a_{n+r+1} + \cdots + c_1 a_{n+2} + c_0 a_{n+1} = c, \qquad n \geq 0.$$

Taking the difference of these two equations yields

$$c_r a_{n+r+1} + (c_{r-1} - c_r) a_{n+r} + \cdots + (c_0 - c_1) a_{n+1} - c_0 a_n = 0, \qquad n \geq 0,$$

i.e., a non-trivial homogeneous recurrence for $(a_n)_{n\geq 0}$ of order $r+1$.

3.1 Fibonacci numbers

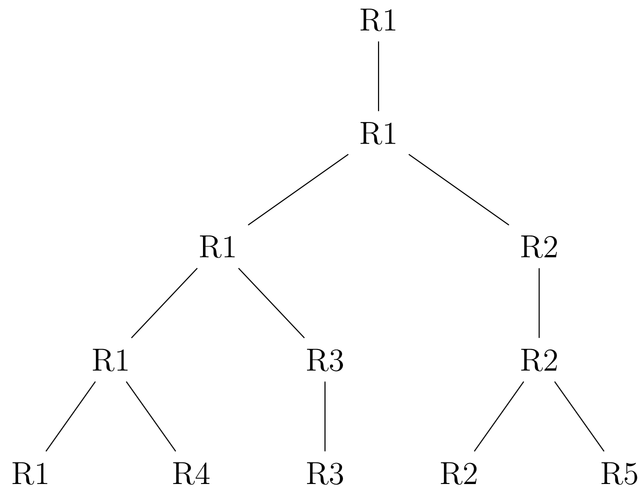
A prominent example of C-finite sequence are Fibonacci numbers. This is in part, because they satisfy a very simple recurrence and in part because of the many applications that these numbers appear in. Typically, they are introduced as

$$F_n = \#(\text{offsprings a single rabbit produces in } n \text{ months}),$$

subject to the following rules:

- the initial rabbit is born in the first month, i.e., $F_0 = 0, F_1 = 1$;
- each rabbit produces 1 rabbit per month starting from the second month of its existence;
- no rabbit ever dies.

Let's have a look at the rabbit population in the first few months. Each level corresponds to a new month, starting from month 1 and Ri denotes rabbit number i.



A recurrence for the Fibonacci numbers is easily derived: we have that at month $n+2$ there are all the rabbits from the previous generation $(n+1)$ and an offspring of rabbits that are at least two months old (n), i.e.,

$$F_{n+2} = F_{n+1} + F_n, \qquad n \geq 0, \quad F_0 = 0, F_1 = 1. \tag{10}$$

From the self-similarity of the tree, we can easily deduce the following functional equation for the generating function $f(x) = \sum_{n\geq 0} F_n x^n$,

$$f(x) = xf(x) + x^2 f(x) + x.$$

Solving for $f(x)$ yields,

$$f(x) = \frac{x}{1-x-x^2} = \sum_{n\geq 0} F_n x^n.$$

Let's return to the recurrence and write (10) as follows,

$$\begin{aligned} F_{n+1} &= F_{n+1} \\ F_{n+2} &= F_{n+1} + F_n \end{aligned}$$

From this we see how to write the recurrence in its matrix form,

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}}_{=:z_n}.$$

Repeated application of the recurrence gives

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1} \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}.$$

The matrix A is called the *companion matrix* of the recurrence and an analogous rewriting in the form

$$z_{n+1} = Az_n = A^{n+1} z_0,$$

can be done for any C-finite recurrence. The Matrix representation can be used to *derive identities* on the sequence. Let's continue with the example of the Fibonacci numbers. First note that for $n = -1$,

$$F_1 = F_0 + F_{-1} \quad \Longrightarrow \quad 1 = 0 + F_{-1},$$

hence the choice $F_{-1} = 1$ is consistent with the recurrence. Then

$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} F_{-1} & F_0 \\ F_0 & F_1 \end{pmatrix}.$$

Taking determinants on both sides yields,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \qquad (n \geq 0), \tag{11}$$

which is known as *Cassini's identity.*

Diagonalization: Let's compute the eigendecomposition of $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, i.e., the matrix S of eigenvectors and the diagonal matrix D of eigenvalues with the property $A = SDS^{-1}$.

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 1 = 0.$$

From this equation we get the two eigenvalues

$$\phi_1 = \frac{1}{2} - \frac{\sqrt{5}}{2}, \quad \text{and} \quad \phi_2 = \frac{1}{2} + \frac{\sqrt{5}}{2} \approx 1.61803\ldots.$$

ϕ_2 is known as the *golden ratio*. The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ \phi_1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ \phi_2 \end{pmatrix}.$$

Hence,

$$S = \begin{pmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{pmatrix} \quad \text{and with} \quad |S| = \phi_2 - \phi_1 = \sqrt{5} \quad \text{we have} \quad S^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{pmatrix}.$$

Then $A = SDS^{-1}$ with $D = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}$ and

$$A^n = (SDS^{-1})^n = SDS^{-1}SDS^{-1} \cdots SDS^{-1} = SD^nS^{-1}.$$

Summarizing, we have,

$$A^n = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}^n \begin{pmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{pmatrix} = \begin{pmatrix} \cdots & \frac{1}{\sqrt{5}}(\phi_2^n - \phi_1^n) \\ \frac{1}{\sqrt{5}}(\phi_2^n - \phi_1^n) & \cdots \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Thus we have derived the *Euler-Binet formula*,

$$F_n = \frac{1}{\sqrt{5}}(\phi_2^n - \phi_1^n), \qquad n \geq 0. \tag{12}$$

Since $\phi_1 \approx -0.61803\ldots$, the asymptotic behaviour of F_n is governed by ϕ_2, i.e.,

$$F_n \sim \frac{1}{\sqrt{5}}\phi_2^n (n \to \infty), \tag{13}$$

hence also

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi_2.$$

A different approach for finding a closed form solution of the Fibonacci recurrence (10) is by considering the *characteristic polynomial* of the recurrence: to this end, replace F_n by x^n in the recurrence:

$$F_{n+2} - F_{n+1} - F_n = 0 \quad \longrightarrow \quad x^{n+2} - x^{n+1} - x^n = x^n(\underbrace{x^2 - x - 1}_{=:\chi(x)}) = 0.$$

The polynomial $\chi(x)$ is called the characteristic polynomial of the recurrence and the roots of it are ϕ_1 and ϕ_2. Then (as can easily be checked),

$$a_n = \alpha\phi_1^n + \beta\phi_2^n, \qquad \forall \alpha, \beta \in \mathbb{K},$$

is a solution to (10).

The initial values yield

$$\begin{aligned} n = 0: &\qquad 0 = \alpha + \beta \\ n = 1: &\qquad 1 = \alpha\phi_1 + \beta\phi_2. \end{aligned}$$

Solving this system gives $\alpha = -\frac{1}{\sqrt{5}}$ and $\beta = \frac{1}{\sqrt{5}}$ and hence we obtained once more the Euler-Binet formula $F_n - \frac{1}{\sqrt{5}}(\phi_2^n - \phi_1^n)$.

Example 38. *(Tower of Hanoi)*

Given: a tower of n disks that are initially stacked in increasing size on one of three pegs.

Task: transfer the entire tower to one of the other pegs moving only one disk at a time and never moving a larger disk onto a smaller one.

Find: a_n = minimal number of moves needed.

Example 39. *n people numbered from 1 to n are sitting at a round table; starting from person 1 in clockwise order every second person leaves until only one person remains. What is $J(n)$ = number of the remaining person?*

3.2 Properties of C-finite recurrences

In general, the closed form of a C-finite recurrence can be derived from its characteristic polynomial.

Theorem 40. *Let $c_0, c_1, \dots, c_{r-1} \in \mathbb{K}$ with $c_0 \neq 0$ be such that*

$$x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0 = (x - \alpha_1)^{m_1} \cdot \dots \cdot (x - \alpha_k)^{m_k}$$

for $m_j \in \mathbb{N}^*$ and pairwise distinct $\alpha_j \in \mathbb{K}$. Then the sequences

$$(n^i \alpha_j^n)_{n\geq 0} \quad \textit{for} \quad 1 \leq j \leq k, 0 \leq i \leq m_j - 1,$$

form a basis of solutions *of the* $\mathbb{K}$-vector space *of all solutions of the recurrence*

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \quad n \geq 0.$$

Example 41. *Let the sequence $(a_n)_{n\geq 0}$ be defined by*

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0, \qquad n \geq 0 \quad \textit{with} \quad a_0 = 0, a_1 = 1, a_2 = 3.$$

Then the characteristic polynomial is

$$\chi(x) = x^3 - 3x^2 + 3x - 1 = (x - 1)^3$$

and it has one triple root $x = 1$. Hence, by the theorem, the general solution is

$$a_n = \gamma_0 1^n + \gamma_1 n 1^n + \gamma_2 n^2 1^n.$$

From the initial values, we compute $\gamma_0 = 0, \gamma_1 = \gamma_2 = \frac{1}{2}$ and thus

$$a_n = \frac{1}{2}n(n+1) = \binom{n+1}{2}.$$

What is the generating function of this sequence, i.e., what is $A(x) = \sum_{n\geq 0} a_n x^n$? Let's start from the recurrence and multiply it by x^{n+3}:

$$a_{n+3}x^{n+3} - 3a_{n+2}x^{n+3} + 3a_{n+1}x^{n+3} - a_n x^{n+3} = 0.$$

Summing over $n \geq 0$ and pulling out some powers of x gives

$$\underbrace{\sum_{n\geq 0} a_{n+3}x^{n+3}}_{A(x) - x - 3x^2} - 3x \underbrace{\sum_{n\geq 0} a_{n+2}x^{n+2}}_{A(x) - x} + 3x^2 \underbrace{\sum_{n\geq 0} a_{n+1}x^{n+1}}_{A(x) - a_0 x^0} - x^3 \underbrace{\sum_{n\geq 0} a_n x^n}_{A(x)} = 0.$$

Putting everything together gives

$$A(x) = \frac{x}{(1-x)^3}.$$

This is again a rational function as it was in the case of the Fibonacci numbers and this is not a coincidence.

Theorem 42. *A sequence $(a_n)_{n\geq 0}$ in $\mathbb{K}$ satisfies a C-finite recurrence*

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \quad n \geq 0,$$

with $c_i \in \mathbb{K}$ and $c_0 \neq 0$ if and only if

$$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_1 x^{r-1} + c_0 x^r}$$

for some $p \in \mathbb{K}[x]$ with $\deg p(x) \leq r - 1$.

Theorem 43. *(Closure properties) Let $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ be C-finite sequences in $\mathbb{K}$ of orders r and s, respectively, and let $m \in \mathbb{N}^*$. Then,*

(1) $(a_n + b_n)_{n\geq 0}$ is C-finite of order at most $r + s$.

(2) $(a_n \cdot b_n)_{n\geq 0}$ is C-finite of order at most rs.

(3) $(\sum_{k=0}^{n} a_k)_{n\geq 0}$ is C-finite of order at most $r + 1$.

(4) $(a_{mn})_{n\geq 0}$ is C-finite of order at most r.

(5) $(a_{\lfloor n/m \rfloor})_{n\geq 0}$ is C-finite of order at most mr.

Proof. (1) By definition there exist $q_0, \ldots, q_r, p_0, \ldots, p_s \in \mathbb{K}$, not all zero, s.t. for all $n \geq 0$,

$$q_r a_{n+r} + \cdots + q_1 a_{n+1} + q_0 a_n = 0$$
$$p_s b_{n+r} + \cdots + p_1 b_{n+1} + p_0 b_n = 0$$

Then $c_n = a_n + b_n$ is in the vector space generated by

$$(a_n)_{n\geq 0}, (a_{n+1})_{n\geq 0}, \ldots, (a_{n+r-1})_{n\geq 0}, (b_n)_{n\geq 0}, (b_{n+1})_{n\geq 0}, \ldots, (b_{n+s-1})_{n\geq 0}.$$

This vector space contains all the shifted sequences $(a_{n+k})_{n\geq 0}, (b_{n+k})_{n\geq 0}$ for $k \in \mathbb{N}$. The dimension of this space is at most $r + s$, hence any $r + s + 1$ sequences $(c_{n+k})_{n\geq 0}$ *must* be linearly dependent. □

This theorem is constructive in the sense that, given the recurrence coefficients of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, the recurrence coefficients of $(a_n + b_n)_{n\geq 0}, (a_n \cdot b_n)_{n\geq 0}, \ldots$ can be computed.

Example 44. *Consider the following two combinatorial sequences given by their defining recurrences: the Lucas numbers* $(L_n)_{n\geq 0}$*,*

$$L_{n+2} - L_{n+1} - L_n = 0, \qquad L_0 = 2, L_1 = 1, \quad n \geq 0,$$

and the Perrin numbers,

$$P_{n+3} - P_{n+1} - P_n = 0, \qquad P_0 = 3, P_1 = 0, P_2 = 2, \quad n \geq 0.$$

For $n \geq 0$*, let* $a_n = L_n + P_n$*. According to Theorem 43,* $(a_n)_{n\geq 0}$ *satisfies a recurrence of order at most five. This recurrence can be computed using an ansatz solving for undetermined coefficients.*

$$\begin{aligned}
a_n &= L_n + P_n \\
a_{n+1} &= L_{n+1} + P_{n+1} \\
a_{n+2} &= L_{n+2} + P_{n+2} = L_{n+1} + L_n + P_{n+1} \\
a_{n+3} &= L_{n+2} + L_{n+1} + P_{n+3} = 2L_{n+1} + L_n + P_{n+1} + P_n \\
a_{n+4} &= 3L_{n+1} + 2L_n + P_{n+2} + P_{n+1} \\
a_{n+5} &= 5L_{n+1} + 3L_n + P_{n+2} + P_{n+1} + P_n
\end{aligned}$$

If we define the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 & 5 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

we can write

$$(a_n, a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}, a_{n+5}) = (L_n, L_{n+1}, P_n, P_{n+1}, P_{n+2}) \cdot A.$$

Then for any vector $v \in \mathcal{N}(A) = \{w \mid Aw = 0\}$*, the right nullspace of* A*, we have*

$$(a_n, a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}, a_{n+5})v = (L_n, L_{n+1}, P_n, P_{n+1}, P_{n+2}) \cdot Av = 0.$$

One such (non-trivial) element is $v = (1, 2, 0, -2, -1, 1)^\top$*.*

Example 45. *Let* $s_n = \sum_{k=0}^{n} F_k$ *for* $(F_n)_{n\geq 0}$ *the Fibonacci numbers, i.e.,*

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, F_1 = 1, \quad n \geq 0.$$

Then

$$F_n = s_n - s_{n-1}, \qquad n \geq 1.$$

Plugging this into the Fibonacci recurrence yields,

$$s_{n+2} - s_{n+1} = s_{n+1} - s_n + s_n - s_{n-1}, \qquad n \geq 1,$$

and thus

$$s_{n+3} - 2s_{n+2} + s_n = 0, \qquad s_0 = 0, s_1 = 1, s_2 = 2, \quad n \geq 0.$$

4 Hypergeometric sequences

Definition 46. *A sequence $(a_n)_{n\geq 0}$ is called* hypergeometric *iff there exists a fixed rational function $r \in \mathbb{K}(x)$ s.t.,*

$$a_{n+1} = r(n)a_n, \qquad n \geq 0.$$

We call $r(n)$ the shift quotient *of $(a_n)_{n\geq 0}$.*

Example 47.
- *$a_n = p(n)$ for some $p \in \mathbb{K}[x]$ (*polynomial sequences*)*
- *$a_n = z^n$ for $z \in \mathbb{K}$ or z an indeterminate*
- *$a_n = \alpha^{\overline{n}} = (\alpha)_n$ for $\alpha \in \mathbb{K}$ or α an indeterminate:*

$$\frac{a_{n+1}}{a_n} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\cdot(\alpha+n)}{\alpha(\alpha+1)\cdots(\alpha+n-1)} = \alpha + n.$$

- *products or quotients of any of the above.*

One can unwind the recurrence from definition 46 down to the initial value,

$$a_{n+1} = r(n)a_n = r(n)r(n-1)a_{n-1} = \cdots = r(n)r(n-1)\cdots r(1)r(0)a_0,$$

which gives some sort of closed form solution to a hypergeometric recurrence.

Recall, that solutions of C-finite recurrences are built from sequences of the form,

$$(n^i \alpha^n)_{n\geq 0},$$

for some $\alpha \in \mathbb{K}$ and $i \in \mathbb{N}$ fixed. The shift quotient of this sequence above is

$$\frac{(n+1)^i}{n^i}\frac{\alpha^{n+1}}{\alpha^n} = \left(1 + \frac{1}{n}\right)^i \alpha = r(n) \in \mathbb{K}(n),$$

i.e., solutions of C-finite recurrences are *linear combinations* of hypergeometric terms. These are in general not hypergeometric.

Let's have a closer look at how closure properties work for hypergeometric sequences: let $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ be hypergeometric sequences, i.e.,

$$\exists r, s \in \mathbb{K}(x): \quad \frac{a_{n+1}}{a_n} = r(n) \quad \text{and} \quad \frac{b_{n+1}}{b_n} = s(n).$$

Then:

1. $(a_n b_n)_{n\geq 0}$ is hypergeometric,

$$\frac{a_{n+1}b_{n+1}}{a_n b_n} = r(n)s(n).$$

2. if $a_n \neq 0$ for all $n \geq 0$ then $(1/a_n)_{n\geq 0}$ is hypergeometric.

3. $(a_{kn+d})_{n\geq 0}$ for $k, d \in \mathbb{N}$ *fixed* is hypergeometric.

4. What about $(a_n + b_n)_{n\geq 0}$? We have

$$(a_n + b_n)_{n\geq 0} \text{ hypergeometric} \quad \Leftrightarrow \quad \frac{a_{n+1} + b_{n+1}}{a_n + b_n} = t(n),$$

for some fixed rational function $t \in \mathbb{K}(x)$. Then this is equivalent to

$$\begin{aligned} \frac{r(n)a_n + s(n)b_n}{a_n + b_n} &= t(n) \\ \Leftrightarrow \quad (r(n) - t(n))a_n &= (t(n) - s(n))b_n \\ \Leftrightarrow \quad a_n &= \frac{t(n) - s(n)}{r(n) - t(n)} b_n. \end{aligned}$$

Hence, $(a_n + b_n)_{n\geq 0}$ is hypergeometric if and only if a_n is a rational multiple of b_n.

Definition 48. *Two hypergeometric sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are called* similar *if*

$$\exists p, q \in \mathbb{K}[x]: \qquad p(n)a_n = q(n)b_n, \qquad n \in \mathbb{N}.$$

Example 49.

- *$a_n = n!$ and $b_n = 1$ are not similar.*
- *$a_n = (n + 1)!$ and $b_n = n!$ are similar.*
- *$(a_n)_{n\geq 0}$ and $(a_{n+m})_{n\geq 0}$ ($m \in \mathbb{N}$ fixed) are similar.*

Example 50. *Let*

$$a_{n+2} - 4a_{n+1} + 4a_n = 0, \qquad a_0 = 1, a_1 = 3, \quad n \geq 0.$$

Then the characteristic polynomial is $\chi(x) = x^2 - 4x + 4 = (x - 2)^2$ and thus the general solution to the recurrence is

$$a_n = c_0 2^n + c_1 n 2^n,$$

and from the initial values we compute the particular solution

$$a_n = 2^n + n2^{n-1} = 2^{n-1}(2 + n).$$

Then we have for the shift quotient of this sequence,

$$\frac{a_{n+1}}{a_n} = \frac{2^n(n + 3)}{2^{n-1}(n + 2)} = 2\frac{n + 3}{n + 2},$$

i.e., $(a_n)_{n\geq 0}$ is hypergeometric (and as such, satisfies a first order recurrence with polynomial coefficients

$$(n + 2)a_{n+1} - 2(n + 3)a_n = 0, \qquad a_0 = 1, \quad n \geq 0.$$

Unrolling the recurrence in this form gives

$$\begin{aligned}
a_{n+1} &= 2\frac{n+3}{n+2} \cdot a_n \\
&= 4\frac{(n+3)(n+2)}{(n+2)(n+1)} \cdot a_{n-1} \\
&= \cdots = 2^{n+1}\frac{(n+3)(n+2)\cdots 3}{(n+2)(n+1)\cdots 2} \cdot a_0 \\
&= 2^{n+1}\frac{(3)_{n+1}}{(2)_{n+1}}.
\end{aligned}$$

Since by our assumption $\mathbb{K}$ is algebraically closed, the numerator and denominator of the shift quotient of a hypergeometric sequences can be factored into integer linear factors,

$$\frac{a_{n+1}}{a_n} = c \cdot \frac{(\alpha_1+n)(\alpha_2+n)\cdots(\alpha_p+n)}{(\beta_1+n)(\beta_2+n)\cdots(\beta_q+n)},$$

and thus

$$a_n = c^n \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n} \cdot a_0.$$

Generating functions of hypergeometric sequences are called *hypergeometric series*, commonly noted as follows,

$$ {}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}; x\right) := \sum_{n\geq 0} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{x^n}{n!} \in \mathbb{K}[\![x]\!] \tag{14}$$

with $a_i \in \mathbb{K}$ and $b_i \in \mathbb{K}\setminus\{0,-1,-2,-3,\ldots\}$.

Many elementary series can be expressed in terms of hypergeometric series:

- $$\exp(x) = \sum_{n\geq 0}\frac{x^n}{n!} = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; x\right).$$

- $$\frac{1}{(1+x)^\lambda} = \sum_{n\geq 0}\binom{-\lambda}{n}(-x)^n = \sum_{n\geq 0}\frac{(-\lambda)^{\underline{n}}}{n!}(-x)^n = \sum_{n\geq 0}\frac{(\lambda)_n}{n!}x^n = {}_1F_0\left(\begin{matrix} \lambda \\ - \end{matrix}; x\right).$$

Lastly in this section, we have a look at the *asymptotic behaviour* of hypergeometric sequences. First, recall the definition of the *Gamma function*: for $z \in \mathbb{C}$,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad \text{if} \quad \mathrm{Re}(z) > 0,$$

and else by analytic continuation. We have $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$. In particular, for $n \in \mathbb{N}$, we have

$$\Gamma(n+1) = n!,$$

and

$$(a)_n = a^{\overline{n}} = a \cdot (a+1) \cdots (a+n-1) = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(n+a)}{\Gamma(a)}.$$

The asymptotic behaviour of the gamma function is

$$\Gamma(n+z) \sim \sqrt{2\pi} n^{n+z-\frac{1}{2}} e^{-n} \, (n \to \infty),$$

for $z \in \mathbb{C}$. A special case of this is *Stirling's formula*,

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}.$$

Summarizing, we obtain,

$$\frac{(a)_n}{(b)_n} = \frac{\Gamma(n+a)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(n+b)} \sim \cdots \sim \frac{\Gamma(b)}{\Gamma(a)} n^{a-b} \, (n \to \infty).$$

5 The holonomic universe

Definition 51. *A sequence* $(a_n)_{n\geq 0}$ *is called,* holonomic *or* P-finite*, if there exist polynomials* $p_0, \ldots, p_r \in \mathbb{K}[x]$ *with* $p_0(x) \neq 0$ *and* $p_r(x) \neq 0$ *s.t.,*

$$p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = 0, \qquad \forall n \in \mathbb{N}.$$

If $d = \max_k \deg p_k(x)$*, then we say that* $(a_n)_{n\geq 0}$ *is holonomic of order* r *and degree* d*.*

We will sometimes use the term P-finite, sometimes the term holonomic. Another commonly used notion is *P-recursive*.

Example 52.

- *C-finite sequences are P-finite.*
- *Hypergeometric sequences are P-finite.*
- *Harmonic numbers are P-finite (but not C-finite): Recall that* $H_n = \sum_{k=1}^{n} \frac{1}{k}$ *for* $n \geq 1$ *with* $H_0 = 0$*. We used earlier that*

  $$H_{n+1} - H_n = \frac{1}{n+1},$$

  and hence, for all $n \geq 0$*,*

  $$\begin{aligned} (n+1)H_{n+1} - (n+1)H_n &= 1, \\ (n+2)H_{n+2} - (n+2)H_{n+1} &= 1, \end{aligned}$$

  Subtracting those two equations gives

  $$(n+2)H_{n+2} - (2n+3)H_{n+1} + (n+1)H_n = 0, \qquad n \geq 0, \quad H_0 = 0, H_1 = 1.$$

Remark 53. *An* equivalent *characterization for holonomic sequences is*

$$\exists p_0, \ldots, p_r, q \in \mathbb{K}[x], p_0 \neq 0, p_r \neq 0 \colon p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = q(n), \qquad \forall n \in \mathbb{N}.$$

Also the average number of comparisons in QuickSort is holonomic:

$$(n+1)a(n+1) - (n+2)a(n) = 2n, \qquad n \geq 0, a(0) = 0.$$

Recall that we had the closed form representation,

$$a(n) = 2(n+1)H_n - 4n,$$

which is a combination of multiplying and adding P-finite sequences. The same way as C-finite sequences, also P-finite sequences satisfy several closure properties. We summarize a few in the following theorem.

Theorem 54. *(Closure properties) Let* $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ *be holonomic. Then*

(1) $(\alpha a_n + \beta b_n)_{n\geq 0}$ *is holonomic* ($\alpha, \beta \in \mathbb{K}$ *fixed)*

(2) $\left(\sum_{k=0}^{n} a_k b_{n-k}\right)_{n\geq 0}$ *and* $(a_n \cdot b_n)_{n\geq 0}$ *are holonomic.*

The proofs of these closure properties are analogous to the proofs of the corresponding statements on C-finite sequences and the same bounds on the orders apply. Further closure properties include the forward shift, taking subsequences or interlacing.

Since any hypergeometric sequence is also P-finite, the addition of two hypergeometric sequences is always P-finite (of order at most 2).

The generating function of a holonomic sequence is also called holonomic.

Definition 55. *A formal power series* $a(x) = \sum_{n\geq 0} a_n x^n$ *is called* holonomic *or* D-finite*, iff its coefficient sequence* $(a_n)_{n\geq 0}$ *is holonomic.*

Holonomic power series satisfy *linear differential equations* with polynomial coefficients.

Theorem 56. *A formal power series* $a(x)$ *is* holonomic *if and only if there exist polynomials* $p_0, \ldots, p_r \in \mathbb{K}[x]$*, not all zero, such that*

$$p_r(x)a^{(r)}(x) + \cdots + p_1(x)a'(x) + p_0(x)a(x) = 0.$$

As for sequences, we say that, with $d = \max_k \deg p_k(x)$, the formal power series is holonomic of order r and degree d.

Using operator notation, we can write the ordinary differential equation from the theorem above as

$$\begin{aligned} & p_r(x)D_x^r a(x) + \cdots + p_1(x)D_x a(x) + p_0(x)a(x) = 0 \\ \iff \quad & (p_r(x)\cdot D_x^r + \cdots + p_1(x)\cdot D_x + p_0(x)\cdot 1) \bullet a(x) = 0. \end{aligned}$$

Also for holonomic functions, the equivalent characterization using inhomogeneous differential equations holds.

Theorem 57. *Let* $a(x) = \sum_{n\geq 0} a_n x^n$ *be a formal power series. Then:*

(1) If $a(x)$ *is holonomic of order* r *and degree* d*, then* $(a_n)_{n\geq 0}$ *is holonomic of order at most* $r + d$ *and degree at most* r*.*

(2) If $(a_n)_{n\geq 0}$ *is holonomic of order* r *and degree* d*, then* $a(x)$ *is holonomic of order at most* d *and degree at most* $r + d$*.*

Proof. (1): Suppose $a(x)$ satisfies,

$$p_r(x)D_x^r a(x) + \cdots + p_1(x)D_x a(x) + p_0(x)a(x) = 0,$$

with $\deg p_k(x) \leq d$ for all $k = 0, \ldots, r$ and with $p_0, p_r \neq 0$.

Now,

$$x^i D_x^j a(x) = \sum_{n\geq 0} a_n x^i D_x^j x^n = \sum_{n\geq 0} a_n x^i n^{\underline{j}} x^{n-j} = \sum_{n\geq j} a_n n^{\underline{j}} x^{n+i-j} = \sum_{n\geq i} a_{n-i+j}(n-i+j)^{\underline{j}} x^n.$$

Next, we rewrite the differential equation above,

$$\sum_{j=0}^{r} p_j(x) D_x^j a(x) = \sum_{j=0}^{r}\sum_{i=0}^{d} p_{j,i} x^i D_x^j a(x) = \sum_{n\geq 0} x^n \sum_{j=0}^{r}\sum_{i=0}^{d} p_{j,i} a_{n-i+j} (n-i+j)^{\underline{j}}.$$

Equating the coefficients of x^n to zero for $n \geq d$ gives the recurrence

$$\sum_{j=0}^{r}\sum_{i=0}^{d} p_{j,i} a_{n-i+j} (n-i+j)^{\underline{j}} = 0, \qquad n \geq d.$$

Since j ranges from 0 to r, we have that the degree of the recurrence coefficients is at most r. Concerning the order, note that we have $a_{n-d}, \ldots, a_{n+r}$ appearing in the recurrence above. Hence, we have that the order of the coefficient recurrence is at most $r + d$. □

This proof is constructive, i.e., given the polynomial coefficients of the differential equation, the polynomial coefficients of the recurrence equation can be computed (and vice versa).

Theorem 58. *(Closure properties II) Let $a(x), b(x)$ be holonomic. Then:*

(1) $\alpha a(x) + \beta b(x)$ is holonomic ($\alpha, \beta \in \mathbb{K}$ fixed).

(2) $a(x)b(x)$ is holonomic.

(3) $a'(x), \int_x a(x)$ are holonomic.

Any algebraic formal power series is also holonomic. Recall first that $y(x) \in \mathbb{C}[\![x]\!]$ is *algebraic*, iff

$$\exists q_0, \ldots, q_d \in \mathbb{C}[x], \text{not all zero:} \quad q_0(x) + q_1(x)y(x) + \cdots + q_d(x)y(x)^d = 0.$$

Theorem 59. *If $y(x) \in \mathbb{C}[\![x]\!]$ is algebraic then it is also holonomic.*

The proof of this result is constructive: given the minimal polynomial of degree d, the linear differential equation of order d can be computed.

Theorem 60. *(Algebraic substitution) If $y(x) \in \mathbb{C}[\![x]\!]$ is holonomic and $a(x) \in \mathbb{C}[\![x]\!]$ is algebraic with $a(0) = 0$, then $y(a(x))$ is holonomic.*

Example 61. $\exp\left(\frac{x}{\sqrt{1-4x}}\right) \in \mathbb{C}[\![x]\!]$ *is holonomic, because: $y(x) = \exp(x) \in \mathbb{C}[\![x]\!]$ is holonomic,*

$$y'(x) - y(x) = 0, \quad y(0) = 1,$$

and $a(x) = \frac{x}{\sqrt{1-4x}} \in \mathbb{C}[\![x]\!]$ is algebraic,

$$(1-4x)a(x)^2 - x^2 = 0, \qquad a(0) = 0.$$

Proving the converse, i.e., that a sequence is *non-holonomic*, is often very hard.

Theorem 62. *Let $y(x) \in \mathbb{C}[\![x]\!]$ be holonomic with $y(0) \neq 0$. Then*

$$\frac{1}{y(x)} \quad \textit{holonomic} \quad \Longleftrightarrow \quad \frac{y'(x)}{y(x)} \quad \textit{algebraic.}$$

The can proof can be found, e.g., in Stanley, *Enumerative Combinatorics* [6].

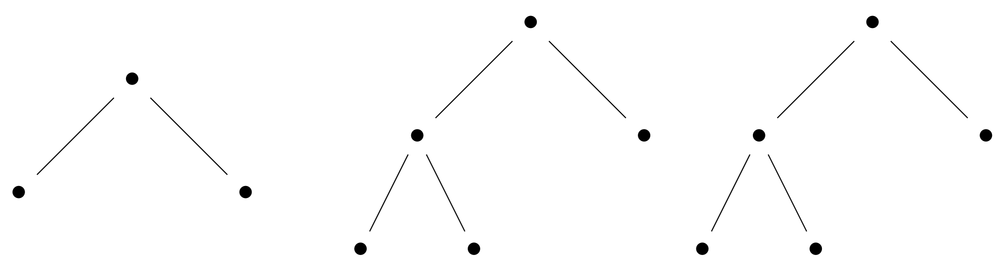
Example 63. *Let $y(x) = \cos(x)$. Since $y(0) = 1 \neq 0$, we can consider the fraction*

$$\frac{y'(x)}{y(x)} = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

The tangent is well known NOT to be algebraic. Hence, by Theorem 62, we conclude that $1/\cos(x)$ is NOT holonomic.

Example 64. *We define a plane binary tree (PBT) as*

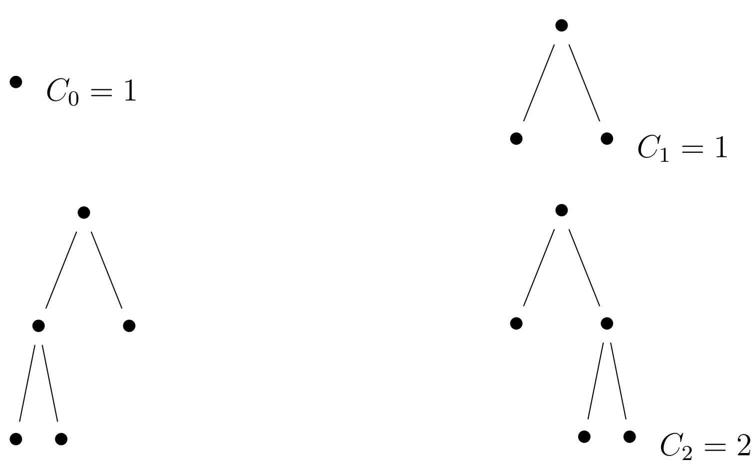
1. *a single node (root) $\bullet$, OR*
2. *the composition of a root with two PBTs.*

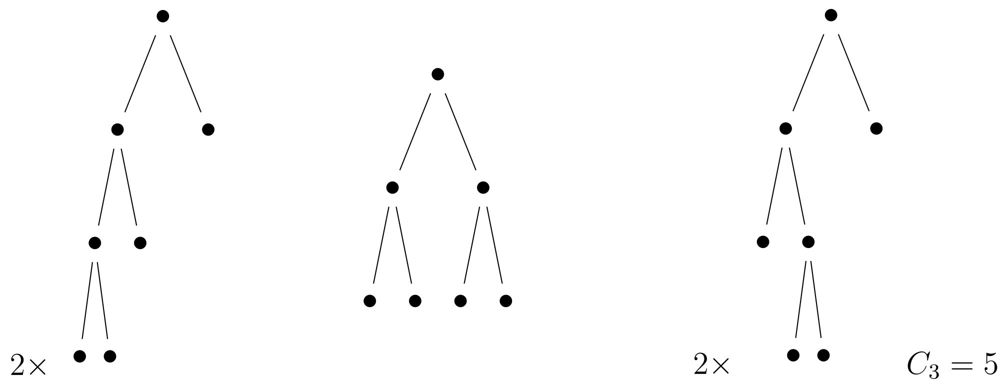


Now let

$$C_n = \#(\textit{PBTs with } n \textit{ internal nodes}).$$

For the first few cases we have:





In general we have that a PBT with n internal nodes is built from 1 root, a subtree with k internal nodes and a subtree with $n-k-1$ nodes. This holds for all $n \geq 1$ and all $0 \leq k \leq n-1$ and can be spelled out as recurrence as follows,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \qquad n \geq 1,\ C_0 = 1.$$

Let's denote the generating function of this sequence by $C(x) = \sum_{n\geq 0} C_n x^n$. Using the recurrence, we have

$$C(x) = 1 + \sum_{n\geq 1}\sum_{k=0}^{n-1} C_k C_{n-k-1} x^n = 1 + x \sum_{n\geq 0}\sum_{k=0}^{n} C_k C_{n-k} x^n,$$

by shifting $n \mapsto n+1$ in the last step. Hence,

$$C(x) = 1 + xC(x)^2,$$

which is an algebraic equation. It can be shown (Exercise!) that

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

The sequence $(C_n)_{n\geq 0}$ is known as the Catalan numbers*.*

6 Polynomial solutions of holonomic recurrences

In this section we derive an algorithm for finding the general polynomial solution (if there is one). In this derivation we restrict ourselves to recurrences of order two for ease of presentation. The generalization to arbitrary orders is immediate. Hence, the problem under consideration is:

Given: $a_0, a_1, a_2 \in \mathbb{K}[x], a_0, a_2 \neq 0, c \in \mathbb{K}[x]$

Find: $y \in \mathbb{K}[x]$ s.t.

$$a_2(n)y(n+2) + a_1(n)y(n+1) + a_0(n)y(n) = c(n), \qquad (n \in \mathbb{N}). \tag{15}$$

Notation

- *forward shift* S_n:
$$S_n f(n) = f(n+1);$$
it is non-commutative w.r.t. n, i.e.,
$$S_n(nf(n)) = (n+1)f(n+1) = (n+1)S_n f(n),$$
or in operator notation: $S_n n = (n+1)S_n$.
- *forward difference* $\Delta_n = S_n - 1$, i.e., $\Delta_n f(n) = f(n+1) - f(n)$.

Note, that

$$\begin{aligned}
\Delta_n n^{\underline{k}} &= \Delta_n \left(n(n-1)\cdots(n-k+1)\right)\\
&= (n+1)\cdot n\cdots(n-k+2) - n\cdot(n-1)\cdots(n-k+2)(n-k+1)\\
&= n^{\underline{k-1}}(n+1-n+k-1)\\
&= kn^{\underline{k-1}}.
\end{aligned}$$

In other words, the action of the forward difference on the falling factorial basis corresponds to the action of the derivative on the monomial basis. Furthermore, for any polynomial p, we have

$$\deg p(n) = d \qquad \Longrightarrow \qquad \deg \Delta_n(p(n)) \leq d-1.$$

Note, that we follow the convention that $\deg 0 = -\infty$.

STEP 1: Determine a *degree bound* D for potential polynomial solutions y of (15).

First, we rewrite the LHS of the recurrence (15) in operator notation and expand in Δ_n (instead of $S_n = \Delta_n + 1$):

$$\begin{aligned}
L = \sum_{i=0}^{2} a_i(n) S_n^i &= \sum_{i=0}^{2} a_i(n)(\Delta_n + 1)^i\\
&= a_0(n) + a_1(n)(\Delta_n + 1) + a_2(n)(\Delta_n^2 + 2\Delta_n + 1)\\
&= (a_0(n) + a_1(n) + a_2(n)) + (a_1(n) + 2a_2(n))\Delta_n + a_2(n)\Delta_n^2.
\end{aligned}$$

For general order r, we have

$$
\begin{aligned}
L = \sum_{i=0}^{r} a_i(n) S_n^i &= \sum_{i=0}^{r} a_i(n)(\Delta_n + 1)^i \\
&= \sum_{i=0}^{r} a_i(n) \sum_{j=0}^{i} \binom{i}{j} \Delta_n^j \\
&= \sum_{j=0}^{r} \left(\sum_{i=j}^{r} \binom{i}{j} a_i(n) \right) \Delta_n^j.
\end{aligned}
$$

Now we define the coefficients in this expansion as

$$b_j(n) = \sum_{i=j}^{r} \binom{i}{j} a_i(n),$$

and then we can rewrite in particular (15) as

$$b_0(n)y(n) + b_1(n)\Delta_n y(n) + b_2(n)\Delta_n^2 y(n) = c(n). \tag{16}$$

Let $d = \deg y(n)$, then we have

$$\deg \Delta_n y(n) \leq d - 1 \quad \text{and} \quad \deg \Delta_n^i y(n) \leq d - i.$$

Define

$$\beta \coloneqq \max_{0 \leq i \leq 2} (\deg b_i(n) - i) \qquad (\beta \coloneqq \max_{0 \leq i \leq r} (\deg b_i(n) - i) \quad \text{in general}).$$

Note, that β can be negative. With this definition, we obviously have that

$$\deg(Ly(n)) \leq d + \beta.$$

Now we distinguish the following cases:

Case 1) $d + \beta < 0$: then a candidate for the degree bound D is

$$D = -\beta - 1 \geq d.$$

Case 2a) $\deg(Ly(n)) = \deg(c(n))$ and $\deg(Ly(n)) = d + \beta$: then a candidate for D is

$$D = \deg(c(n)) - \beta.$$

Case 2b) $\deg(Ly(n)) = \deg(c(n))$ and $\deg(Ly(n)) < d + \beta$: This happens if (at least) the *leading coefficient* of $Ly(n)$ vanishes.

Say $y(n) = \sum_{j=0}^{d} y_j n^{\underline{j}}$. Then

$$\Delta_n y(n) = \sum_{j=0}^{d} y_j j n^{\underline{j-1}} \quad \Longrightarrow \quad \mathrm{lc}(\Delta_n y(n)) = d y_d = d^{\underline{1}} y_d,$$

and

$$\Delta_n^2 y(n) = \sum_{j=0}^{d} y_j j(j-1) n^{\underline{j-2}} \quad \Longrightarrow \quad \mathrm{lc}(\Delta_n^2 y(n)) = d^{\underline{2}} y_d,$$

and so forth. Hence,

$$\mathrm{lc}\left(b_i(n)\Delta_n^i y(n)\right) = \mathrm{lc}(b_i(n))\,\mathrm{lc}(y(n)) d^{\underline{i}}.$$

Thus, we have

$$[n^{d+\beta}]Ly(n) = \sum_{i=0, \deg(b_i)-i=\beta}^{2} \mathrm{lc}(y(n)) d^{\underline{i}}\, \mathrm{lc}(b_i(n)) =: \varphi(d).$$

This is a second degree polynomial in d (in general, a degree r polynomial). Let d_1 denote the *maximal integer root* of this polynomial (with $d_1 = -\infty$ if there is no integer root). This is the final candidate for D,

$$D = d_1 = \max\{m \in \mathbb{N} \mid \varphi(m) = 0\}.$$

STEP 2: Let $D = \max\{-\beta - 1, \deg(c(n)) - \beta, d_1\}$ and set up the ansatz

$$y(n) = \sum_{j=0}^{D} y_j n^j,$$

with undetermined coefficients y_j. Plug this ansatz into (16) and solve for y_j by equating like powers of n on both sides.

Either a polynomial solution is found OR not. In the latter case, we have shown that NO polynomial solution to (15) exists. We summarize these steps in the following algorithm.

Algorithm 65. ***(POLY)***

IN: $a_i \in \mathbb{K}[x], c \in \mathbb{K}[x], i = 0, \ldots, r$

OUT: the general polynomial solution of

$$a_r(n)y(n+r) + \cdots + a_1(n)y(n+1) + a_0(n)y(n) = c(n), \tag{17}$$

if exists.

1. *Compute* $b_j(n) = \sum_{i=j}^{r} \binom{i}{j} a_i(n)$ *for* $j = 0, \ldots, r$

2. $\beta = \max_{0\le i\le r}(\deg(b_i(n)) - i)$

3. $\varphi(d) = \sum_{i=0,\deg(b_i)-i=\beta}^{r} d^{\underline{i}}\,\mathrm{lc}(b_i(n))$

4. $d_1 = \max\{m \in \mathbb{N} \mid \varphi(m) = 0\}$

5. $D = \max\{-\beta - 1, \deg(c) - \beta, d_1\}$

6. *Use the method of undetermined coefficients to find*

$$y(n) = \sum_{j=0}^{D} y_j n^j$$

7. *RETURN* $y(n)$ *OR "No polynomial solution exists"*

Example 66. *We consider the following recurrence,*

$$(n-1)y(n+2) - (3n-2)y(n+1) + 2ny(n) = 0.$$

Hence, we have $c(n) = 0$ *and* $\deg(c(n)) = -\infty$. *The annihilating operator for* $y(n)$ *is*

$$L = (n-1)S_n^2 - (3n-2)S_n + 2n.$$

First rewrite L *in terms of* Δ_n*:*

$$\begin{aligned} L &= (n-1)(\Delta_n + 1)^2 - (3n-2)(\Delta_n + 1) + 2n \\ &= (n-1)\Delta_n^2 + (2n - 2 - 3n + 2)\Delta_n + (n - 1 - 3n + 2 + 2n) \\ &= (n-1)\Delta_n^2 - n\Delta_n + 1, \end{aligned}$$

i.e., $b_2(n) = n-1, b_1(n) = -n, b_0(n) = 1$. *With this we have*

$$\beta = \max_i(\deg(b_i) - i) = \max\{1-2, 1-1, 0-0\} = 0.$$

Then we compute d_1 *as the maximal integer root of*

$$\varphi(d) = \sum_{i=0,\deg b_i - i = 0}^{2} \mathrm{lc}(b_i) d^{\underline{i}} = -d^{\underline{1}} + 1d^{\underline{0}} = -d + 1,$$

hence $d_1 = 1$. *Summarizing, we obtain the degree bound*

$$D = \max\{-\beta - 1, \deg(c) - \beta, d_1\} = \max\{-1, -\infty, 1\} = 1.$$

We set up the ansatz $y(n) = y_0 + y_1 n$ *and plug into the given recurrence:*

$$(n-1)(y_0 + y_1(n+2)) - (3n-2)(y_0 + y_1(n+1)) + 2n(y_0 + y_1 n) = 0.$$

Next we do coefficient comparison (in this case, equate the coefficients to zero):

$$\begin{aligned} &[n^2]LHS = y_1 - 3y_1 + 2y_1 = 0 \qquad \Longrightarrow \qquad 0 = 0 \\ &[n^1]LHS = y_0 + 2y_1 - y_1 - 3y_0 - 3y_1 + 2y_1 + 2y_0 = 0 \qquad \Longrightarrow \qquad 0 = 0 \\ &[n^0]LHS = -y_0 - 2y_1 + 2y_0 + 2y_1 = y_0 \qquad \Longrightarrow \qquad y_0 = 0. \end{aligned}$$

Hence $y(n) = y_1 n$ *is a general polynomial solution to the recurrence.*

7 Summation

7.1 Polynomial sequences

Fix $p \in \mathbb{K}[x]$ with $\deg p(x) = d$. Then $(a_n)_{n\geq 0}$ with

$$a_n = p(n), \quad n \geq 0,$$

is called a *polynomial sequence of degree d*.

Recall that

$$\Delta_n n^{\underline{k}} = k n^{\underline{k-1}} \quad \text{and} \quad \Delta_n^s n^{\underline{k}} = k^{\underline{s}} n^{\underline{k-s}}.$$

Since $\deg p(x) = d$, we have $\Delta_n^{d+1} p(n) = 0$, i.e.

$$\Delta_n^{d+1} p(n) = (S_n - 1)^{d+1} p(n) = \sum_{k=0}^{d+1} \underbrace{\binom{d+1}{k}}_{\in\mathbb{N}} p(n+k) = 0.$$

Hence, $a_n = p(n)$ is C-finite of order $d+1$ (and of course hypergeometric, i.e., holonomic of order one).

Earlier, we had

$$\sum_{k=0}^{d} (2k+1) = \sum_{k=0}^{n} (k+1)^2 - k^2 = (n+1)^2.$$

Now we know how to do *telescoping* for polynomial sequences in a more *systematic* way:

$$k^{\underline{d}} = \frac{1}{d+1} \Delta_k k^{\underline{d+1}}.$$

Hence

$$\sum_{k=0}^{n} k^{\underline{d}} = \frac{1}{d+1} \sum_{k=0}^{n} \left((k+1)^{\underline{d+1}} - k^{\underline{d+1}} \right) = \frac{1}{d+1} (n+1)^{\underline{d+1}}.$$

Example 67. *Since*

$$k^3 = \sum_{j=0}^{3} S_2(3,j) k^{\underline{j}} = 0 \cdot k^{\underline{0}} + 1 \cdot k^{\underline{1}} + 3 \cdot k^{\underline{2}} + 1 \cdot k^{\underline{3}},$$

we have

$$\begin{aligned}\sum_{k=0}^{n} k^3 &= \sum_{k=0}^{n} k^{\underline{1}} + 3 \sum_{k=0}^{n} k^{\underline{2}} + \sum_{k=0}^{n} k^{\underline{3}} \\ &= \tfrac{1}{2}(n+1)^{\underline{2}} + \tfrac{3}{3}(n+1)^{\underline{3}} + \tfrac{1}{4}(n+1)^{\underline{4}} = \cdots = \left(\tfrac{n(n+1)}{2}\right)^2.\end{aligned}$$

We know that $\sum_{k=0}^{n} k = \binom{n}{2} = \frac{n(n+1)}{2}$ and hence, we have that

$$\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k \right)^2.$$

This is an identity that can also be illustrated nicely using a "pebble" argument.

Remark 68. *It follows that for any polynomial p of degree d, the sum*

$$\sum_{k=0}^{n} p(k) = q(k)$$

is some polynomial q of degree $d+1$. Hence, we can use interpolation to sum polynomials.

Remark 69. *Polynomial sequences are closed under the same operations as C-finite sequences.*

7.2 C-finite sequences

We know that if $(a_n)_{n\geq 0}$ is C-finite of order r, then the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ is C-finite of order $r+1$.

Example 70. *Let $s_n = \sum_{k=0}^{n} F_k$, then $s_{n+1} - s_n = F_{n+1}$ and plugging this into the Fibonacci recurrence*

$$F_{n+2} - F_{n+1} - F_n = 0, \qquad F_0 = 0, F_1 = 1, \quad (n \geq 0)$$

gives the order three recurrence

$$s_{n+3} - 2s_{n+2} + s_n = 0, \qquad s_0 = 0, s_1 = 1, s_2 = 2, \quad (n \geq 0).$$

Then we can compute the characteristic polynomial of this recurrence,

$$\chi(x) = x^3 - 2x^2 + 1 = (x^2 + x - 1)(x - 1),$$

from which we know how to compute a closed form solution.

Note that

$$s_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} a_k b_{n-k}$$

for the constant sequence $b_n = 1$. Hence, with $a(x) = \sum_{n\geq 0} a_n x^n$, we have

$$\sum_{n\geq 0} s_n x^n = \frac{1}{1-x} a(x).$$

More interesting is the question: Can we express $s_n = \sum_{k=0}^{n} a_k$ in terms of a_n? For example, for the Fibonacci numbers we have

$$\sum_{k=0}^{n} F_k = F_{n+1} + F_n - 1, \qquad n \geq 0.$$

Instead of using $s_{n+1} - s_n = a_n$ we ant to find b_k, a linear combination of a_k and shifts of a_k, with

$$a_k = b_{k+1} - b_k.$$

Example 71.

$$F_{k+1} = F_{k+1} + \underbrace{F_{k+2} - F_{k+1} - F_k}_{=0} = (F_{k+2} + F_{k+1}) - (F_{k+1} + F_k).$$

Thus

$$\begin{aligned}\sum_{k=0}^{n} F_k = \sum_{k=1}^{n} F_k &= \sum_{k=0}^{n-1} F_{k+1} \\ &= \sum_{k=0}^{n-1} (F_{k+2} + F_{k+1}) - (F_{k+1} + F_k) \\ &= (F_{n+1} + F_n) - (F_1 + F_0) = F_{n+1} + F_n - 1.\end{aligned}$$

This derivation was a bit ad hoc, but it is possible to derive such identities in a systematic way. For this, we define the following operation:

$$\otimes\colon \mathbb{K}[x] \times \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}, \quad (c_0 + c_1 x + \cdots + c_r x^r) \otimes (a_n)_{n\geq 0} := (c_0 a_n + c_1 a_{n+1} + \cdots + c_r a_{n+r})_{n\geq 0}.$$

Example 72.

$$(x-1) \otimes (a_n)_{n\geq 0} = (a_{n+1} - a_n)_{n\geq 0} = (\Delta_n a_n)_{n\geq 0}$$

Let $(a_n)_{n\geq 0}$ be a C-finite sequence satisfying

$$c_0 a_n + c_1 a_{n+1} + \cdots + a_{n+r} = 0.$$

Then the characteristic polynomial is given by

$$\chi(x) = c_0 + c_1 x + \cdots + x^r$$

and we can distinguish the following two cases:

Case 1: $\chi(1) \neq 0$ Then

$$\chi(x) = (x-1)q(x) + \chi(1)$$

for some

$$q(x) = q_0 + q_1 x + \cdots + q_{r-2} x^{r-2} + x^{r-1}.$$

Define

$$(b_n)_{n\geq 0} := q(x) \otimes (a_n)_{n\geq 0} = (q_0 a_n + q_1 a_{n+1} + \cdots + a_{n+r-1})_{n\geq 0}.$$

Then

$$\begin{aligned} & \chi(x) \otimes (a_n)_{n\geq 0} = 0 \\ \Leftrightarrow \quad & (x-1)q(x) \otimes (a_n)_{n\geq 0} + \chi(1)(a_n)_{n\geq 0} = 0\end{aligned}$$

and thus

$$b_{n+1} - b_n + \chi(1) a_n = 0, \qquad n \geq 0.$$

Using telescoping this yields,

$$\sum_{k=0}^{n} a_k = -\frac{1}{\chi(1)}(b_{n+1} - b_0), \qquad n \geq 0.$$

Case 2: $\chi(1) = 0$ Then

$$\chi(x) = (x-1)^m \bar{\chi}(x)$$

for some $m \geq 1$, $\bar{\chi}(x) \in \mathbb{K}[x]$ with $\bar{\chi}(1) \neq 0$. Then

$$\chi(x) = (x-1)^m \left((x-1)\bar{q}(x) + \bar{\chi}(1)\right).$$

Let $(b_n)_{n\geq 0} = \bar{q}(x) \otimes (a_n)_{n\geq 0}$. Then

$$\Delta^m(b_{n+1} - b_n + \bar{\chi}(1)a_n) = 0, \qquad n \geq 0.$$

The m-fold differences can be undone by repeated summation.

For the first sum, we have

$$\sum_{k=0}^{n} \Delta_k \left(\underbrace{\Delta_k^{m-1}(b_{k+1} - b_k + \bar{\chi}(1)a_k)}_{=F(k)} \right) = 0.$$

By telescoping, we obtain $F(n+1) - F(n) = 0$, hence $F(n) = \text{const}$. The second sum shows that $\Delta_k^{m-2}(b_{k+1} - b_k + \bar{\chi}(1)a_k)$ must be a *linear* polynomial. After m summations, we have an identity of the form

$$b_{k+1} - b_k + \bar{\chi}(1)a_k = \bar{p}(k), \qquad k \geq 0,$$

for some polynomial $\bar{p}(k)$ of degree at most $m-1$. Hence, in this case, summing a C-finite sequence amounts to *telescoping* and *summing a polynomial sequence.*

Example 73. *Let $(a_n)_{n\geq 0}$ be given by the recurrence*

$$a_{n+2} - 6a_{n+1} + 9a_n = 0, \qquad a_0 = -1, a_1 = 3.$$

For the characteristic polynomial we have

$$\chi(x) = x^2 - 6x + 9 \qquad \textit{and} \qquad \chi(1) = 4 \neq 0.$$

Polynomial division yields

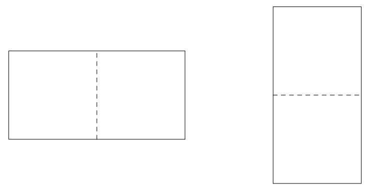
$$\chi(x) = (x-1)(x-5) + 4, \quad \textit{i.e.} \quad q(x) = x - 5.$$

Hence, we define $b_n = a_{n+1} - 5a_n$ and we have for the partial sum

$$\sum_{k=0}^{n} a_k = -\frac{1}{4}\left(a_{n+2} - 5a_{n+1} - a_1 + 5a + 0\right) = -\frac{1}{4}\left(a_{n+2} - 5a_{n+1} - 8\right).$$

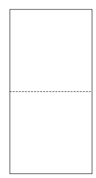
Before we turn to summation for the next class of sequences (which will be hypergeometric sequences), we present another example for a C-finite sequence.

Example 74. *Let T_n denote the number of ways to cover a $2 \times n$ rectangle with 2×1 dominoes, i.e.,*

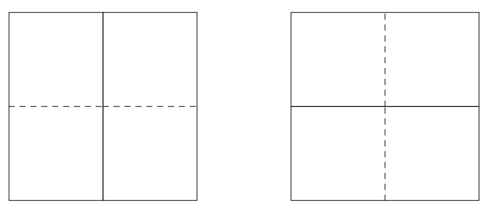


Before we can derive a recurrence for the general pattern, as usual we have a look at some particular cases:

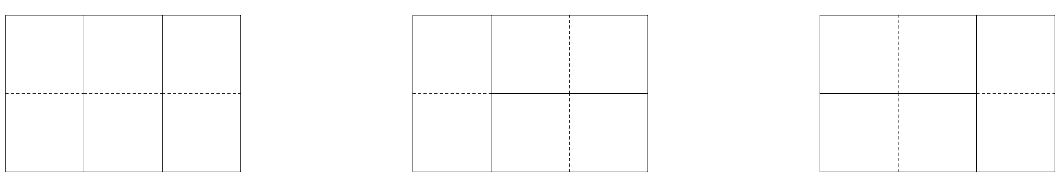
- $T_0 = 1$*: there is only one way to cover the empty rectangle*
- $T_1 = 1$*: there is only one way to cover a 2×1 rectangle*



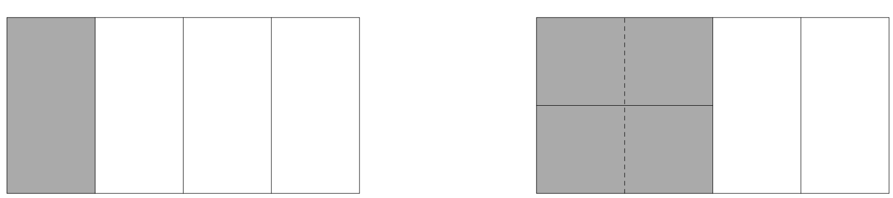
- $T_2 = 2$*: there are the following two ways to cover a 2×2 rectangle*



- $T_3 = 3$*: there are the following three ways to cover a 2×3 rectangle*



- $T_4 =?$*: there are two ways to cover the first column,*



depending on which there is either a 2×3 or a 2×2 rectangle left to be covered. Hence $T_4 = T_3 + T_2 = 5$.

In general: there are two ways to cover the first column and then one is left with one $2 \times (n-1)$ *and one* $2 \times (n-2)$ *rectangle, hence,*

$$T_n = T_{n-1} + T_{n-2}, \qquad T_0 = T_1 = 1.$$

These are just the Fibonacci numbers - compared to our convention, shifted by one to the left. Thus the generating function now is given by

$$f(z) = \sum_{n \geq 0} T_n z^n = \frac{1}{1 - z - z^2}.$$

Instead of just counting the overall number of tiles, we could also count the number of horizontal and vertical tiles separately and consider

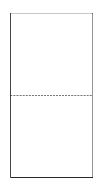
$$c(i,j) = \#(\text{ ways to cover a } 2 \times (i+j) \text{ rectangle with } i \text{ vertical and } j \text{ horizontal tiles})$$

This can be encoded in a bivariate generating function

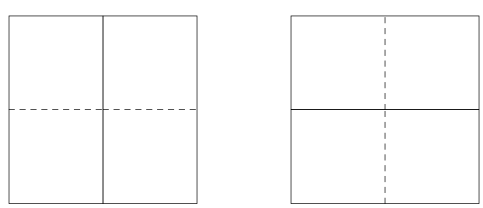
$$F(x,y) = \sum_{i,j \geq 0} c(i,j) x^i y^j,$$

where x *encodes the vertical and* y *the horizontal tiles. Then the coverings of different rectangles that we considered earlier correspond to the following encodings:*

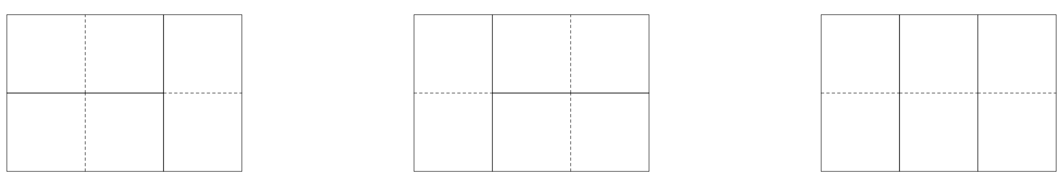
- $T_1 = 1$ *corresponds to* $1 \cdot x^1 y^0$



- $T_2 = 2$ *corresponds to* $1 \cdot x^0 y^2 + 1 \cdot x^2 y^0$



- $T_3 = 3$ *corresponds to* $2 \cdot x^1 y^2 + 1 \cdot x^3 y^0$



Observation: the horizontal pieces can only appear in even numbers, i.e.,

$$F(x,y) = \sum_{i,j\geq 0} c(i,2j)x^i y^{2j}.$$

Obviously we have that all possible combinations are given by

$$c(i,2j) = \binom{i+j}{i}$$

and hence

$$F(x,y) = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^{2j} = \sum_{i,k\geq 0} \binom{k}{i} x^i y^{2k-2i} = \sum_{k\geq 0} (x+y^2)^k = \frac{1}{1-x-y^2}.$$

Back to counting tiles,

$$T_n = \#(\text{ways to cover a } 2\times n \text{ rectangle with } 2\times 1 \text{ tiles}).$$

In the generating function this corresponds to setting $x = y = z$, i.e.,

$$f(z) = \sum_{n\geq 0} T_n z^n = F(z,z) = \sum_{i,k\geq 0} \binom{i+k}{i} z^{i+2k}.$$

Then $[z^n]F(z,z) = T_n$ and thus

$$T_n = \sum_{i+2k=n} \binom{i+k}{i} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-2k+k}{n-2k} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{n-2k}.$$

Now, since $T_n = F_{n+1}$, we found another summation identity, namely,

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{n-2k}.$$

7.3 Hypergeometric sequences

The main question of this section is: When is the sum over a hypergeometric term again hypergeometric? We know that, if $(a_n)_{n\geq 0}$ is hypergeometric, this does not imply that the partial sum sequence

$$(s_n)_{n\geq 0} \quad \text{with} \quad s_n = \sum_{k=0}^{n} a_k, \quad n \geq 0$$

is again hypergeometric.

Example 75. *The sequence $a_k = \frac{1}{k+1}$ is hypergeometric, but the sequence of harmonic numbers*

$$H_n = \sum_{k=0}^{n-1} a_k = \sum_{k=1}^{n} \frac{1}{k}$$

is not hypergeometric (for instance because $H_n \sim \log n (n \to \infty)$).

But we will show that for some hypergeometric sequences $(a_k)_{k\geq 0}$, we can find a hypergeometric sequence $(b_k)_{k=0}$ with $a_k = b_{k+1} - b_k$. In that case, we have for the partial sums

$$s_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (b_{k+1} - b_k) = b_{n+1} - b_0,$$

i.e., s_n is (at least) hypergeometric plus a constant.

Example 76. *Let $a_k = k \cdot k!$. Then $b_k = k!$ is a* hypergeometric telescoper *for a_k as*

$$b_{k+1} - b_k = (k+1)! - k! = (k+1-1)k! = k \cdot k!,$$

and hence,

$$s_n = \sum_{k=0}^{n} k \cdot k! = (n+1)! - 1.$$

We refer to sums of the type mentioned so far as *indefinite summation* in the sense that the summand does not depend on the summation bound. Summarizing, we are looking at the following problem:

Given $(a_k)_{k\geq 0}$ hypergeometric

Find $(b_k)_{k\geq 0}$ hypergeometric s.t.

$$a_k = b_{k+1} - b_k, \tag{18}$$

if such a b_k exists.

We will refer to equation (18) as the *telescoping equation*. Next, we go through the derivation of an algorithm to solve this problem, *Gosper's algorithm*.

Since $(a_k)_{k\geq 0}$ is hypergeometric, there exists a fixed rational function $r \in \mathbb{K}(x)$ with $a_{k+1} = r(k)a_k$ (almost everywhere: since there can only be finitely many zeros in the denominator of r, we assume for sake of simplicity that forward shift quotient is defined for all $k \geq 0$).

Analogously, if $(b_k)_{k\geq 0}$ is hypergeometric, then there exists a rational function $s \in \mathbb{K}(x)$ with $b_{k+1} = s(k)b_k$.

Hence, by the telescoping equation (18),

$$a_k = b_{k+1} - b_k = (s(k) - 1)b_k.$$

Since $s(k) - 1$ is rational, this implies that a_k and b_k have to be similar, see Definition 48. Let

$$w(k) = \frac{1}{s(k) - 1} \quad \Rightarrow \quad b_k = w(k)a_k.$$

If we plug this into (18), we obtain

$$a_k = b_{k+1} - b_k = w(k+1)a_{k+1} - w(k)a_k = w(k+1)r(k)a_k - w(k)a_k,$$

and after diving through a_k, we end up with $1 = w(k+1)r(k) - w(k)$. This way, we arrived at the new *equivalent* task:
Find $w \in \mathbb{K}(x)$ s.t.

$$1 = w(x+1)r(x) - w(x). \tag{19}$$

So in the first step, we reduced the problem of finding a hypergeometric solution to the problem of finding a rational solution.

The numerator and the denominator on both sides of (19) have to be equal. Let

$$r(x) = \frac{r_1(x)}{r_2(x)} \quad \text{and} \quad w(x) = \frac{w_1(x)}{w_2(x)},$$

for polynomials $r_i, w_i \in \mathbb{K}[x]$ $(i = 1, 2)$ with

$$\gcd(r_1(x), r_2(x)) = \gcd(w_1(x), w_2(x)) = 1.$$

Plugging this into (19) we get

$$1 = \frac{w_1(x+1)r_1(x)w_2(x) - w_1(x)w_2(x+1)r_2(x)}{r_2(x)w_2(x)w_2(x+1)}.$$

Since on the left hand side, we have a number, some cancellation has to happen on the right hand side. In other words, we must have

$$\begin{aligned}
& w_2(x) | w_1(x+1)r_1(x)w_2(x) - w_1(x)w_2(x+1)r_2(x) \\
\Rightarrow \quad & w_2(x) | w_1(x)w_2(x+1)r_2(x) \qquad (\gcd(w_1(x), w_2(x) = 1) \\
\Rightarrow \quad & w_2(x) | w_2(x+1)r_2(x) && (20)
\end{aligned}$$

and

$$
\begin{aligned}
& w_2(x+1) | w_1(x+1) r_1(x) w_2(x) - w_1(x) w_2(x+1) r_2(x) \\
\Rightarrow \quad & w_2(x+1) | w_1(x+1) r_1(x) w_2(x) \qquad (\gcd(w_1(x), w_2(x) = 1) \\
\Rightarrow \quad & w_2(x) | r_1(x-1) w_2(x-1).
\end{aligned} \tag{21}
$$

The two criteria (20) and (21) give an idea how $w_2(x)$ has to look like: Let $p(x)$ be an *irreducible factor* of $w_2(x)$. Then from (21) we get

$$p(x) | r_1(x-1) \quad \text{OR} \quad p(x) | w_2(x-1).$$

In the latter case, we have $p(x+1)|w_2(x)$ and we have found another irreducible factor of $w_2(x)$. Then again by (21) we have

$$p(x+1) | r_1(x-1) \quad \text{OR} \quad p(x+1) | w_2(x-1).$$

In the latter case, we can argue as before. Since w_2 is a polynomial of finite degree, this process has to *terminate*. Hence,

$$\exists i \in \mathbb{N}: \quad p(x+i) | r_1(x-1).$$

Analogously, using (20), one can derive

$$\exists j \in \mathbb{N}: \quad p(x-j) | r_2(x).$$

Summarizing, we found that $w(x)$ can have a *non-trivial denominator* ONLY if for some $h \in \mathbb{N}^*$,

$$\gcd(r_1(x), r_2(x+h)) \neq 1.$$

The next goal is to reduce the problem further to finding *polynomial solutions*. Based on the observations above *William R. Gosper* suggested to use the following form of r,

$$r(x) = \frac{t(x+1)}{t(x)} \frac{u(x)}{v(x+1)} \quad \text{with} \quad \gcd(u(x), v(x+h)) = 1, \quad h \geq 1. \tag{22}$$

This representation is now known as the *Gosper form*. Every rational function can be written in this way. If we substitute (22) into (19), we obtain

$$\underbrace{w(x+1)t(x+1)}_{=\tilde{w}(x+1)} \frac{u(x)}{v(x+1)} - \underbrace{w(x)t(x)}_{=\tilde{w}(x)} = t(x).$$

This is an equation of the same form as (19) with a polynomial right hand side. Since

$$\gcd(u(x), v(x+h)) = 1 \qquad \text{for} \quad h \geq 1,$$

by the reasoning above, we can conclude that any solution $\tilde{w}$ of this equation has a *trivial denominator*, i.e., $\mathrm{den}(\tilde{w}(x)) \in \mathbb{K}$. A similar argument shows that $v(x+1)$ has to cancel $\tilde{w}(x+1)$ and we arrive at the final substitution

$$\tilde{w}(x) = v(x)y(x) \in \mathbb{K}[x].$$

This completes the reduction from searching for a hypergeometric solution to searching for a polynomial solution of the *Gosper equation*:
Find $y \in \mathbb{K}[x]$:

$$u(x)y(x+1) - v(x)y(x) = t(x). \tag{23}$$

The algorithm POLY (Algorithm 65) *decides* whether this equation has a solution. Hence, also the algorithm we just derived *decides* whether a hypergeometric telescoper exists.

Algorithm 77. ***(GOSPER's ALGORITHM, 1978)***

IN: $(a_k)_{k\geq 0}$ *hypergeometric*

OUT: $(b_k)_{k\geq 0}$ *hypergeometric with* $a_k = b_{k+1} - b_k$ *OR "NO hypergeometric telescoper exists"*

1. *Compute* $r(k) = \frac{a_{k+1}}{a_k}$
2. *Determine the* Gosper form (22) *of* r*:*

$$r(x) = \frac{t(x+1)}{t(x)} \frac{u(x)}{v(x+1)} \quad \textit{with} \quad \gcd(u(x), v(x+h)) = 1, \quad h \geq 1.$$

3. *Use POLY (Algorithm 65) to find* $y \in \mathbb{K}[x]$*:*

$$u(x)y(x+1) - v(x)y(x) = t(x),$$

   if exists.

4. *Either let*

$$w(x) = \frac{v(x)}{t(x)} y(x)$$

   and RETURN $b_k = w(k)a_k$ *OR RETURN "NO hypergeometric antidifference"*

Example 78. *We execute Gosper's algorithm with the input of Example 76, i.e., let* $a_k = k \cdot k!$.

1.

$$r(k) = \frac{(k+1)(k+1)!}{k\, k!} = \frac{(k+1)^2}{k}.$$

2. *We write*
$$r(x) = \frac{x+1}{x}\frac{x+1}{1}$$
and choose
$$t(x) = x, \quad u(x) = x+1, \quad v(x) = 1.$$

3. *The Gosper equation reads as*
$$(x+1)y(x+1) - y(x) = x.$$
The algorithm POLY yields the degree bound $D = 1$ and so we use the ansatz $y(x) = y_0 + y_1 x$. Plugging in the Gosper equation gives,
$$(x+1)(y_0 + y_1 + y_1 x) - (y_0 + y_1 x) = x \quad \Rightarrow \quad x^2 y_1 + x(y_0 + y_1) + y_1 = x.$$
Coefficient comparison yields $y(x) = 1$.

4. *We compute*
$$w(x) = \frac{v(x)}{t(x)} y(x) = \frac{1}{x} \cdot 1,$$
and thus have
$$b_k = w(k) a_k = \frac{1}{k} k\, k! = k!$$
and $a_k = (k+1)! - k!$ as expected.

In this example the polynomials in the Gosper form were chosen by "looking". The question remains, how to compute the Gosper form algorithmically. Given $r \in \mathbb{K}(x)$, we need to determine polynomials t, u, v s.t.
$$r(x) = \frac{t(x+1)}{t(x)} \frac{u(x)}{v(x+1)} \quad \text{with} \quad \gcd(u(x), v(x+h)) = 1, \quad h \geq 1.$$
First, write
$$r(x) = \frac{f(x)}{g(x)} \quad \text{with} \quad f, g \in \mathbb{K}[x], \quad \gcd(f(x), g(x)) = 1.$$
Using *resultants*, possible values for h with $\gcd(f(x), g(x+h)) \neq 1$ can be computed. Let $h \in \mathbb{N}^*$ be one of these candidates (if there are any) and let $z(x)$ be a non-constant common factor of $f(x)$ and $g(x+h)$. Then
$$f(x) = z(x) f_1(x) \quad \text{and} \quad g(x) = z(x-h) g_1(x),$$
for some polynomials $f_1, g_1 \in \mathbb{K}[x]$, and
$$r(x) = \frac{f_1(x)}{g_1(x)} \frac{z(x)}{z(x-h)} \frac{(z(x-1) \cdot \dots \cdot z(x-h+1)}{z(x-h+1) \cdot \dots \cdot z(x-1)}.$$
This way, we have found a first factor
$$t_1(x) = (z(x-1) \cdot \dots \cdot z(x-h+1)$$
of $t(x)$. Repeating this process with f_1, g_1 yields the Gosper form after *finitely* many steps.

7.4 Zeilberger's algorithm

Let

$$s(n,a) = \sum_{k=0}^{n} \binom{a}{k} = \sum_{k=0}^{n} \frac{a^{\underline{k}}}{k!},$$

for $n \geq 0$ and a a formal parameter. If we run Gosper's algorithm on this input, the output is "no solution", but we know that $s(n,n)$ has a simple closed form,

$$s(n,n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad n \geq 0.$$

This type of summand is not in the scope of Gosper's algorithm, as the summation bound occurs in it. We refer to this type of sums as *definite sums*. In this section, we consider particular definite sums with summands that are *hypergeometric in both* n and k, i.e.,

$$\exists r_1, r_2 \in \mathbb{K}(x,y)\colon \quad \frac{a(n+1,k)}{a(n,k)} = r_1(n,k) \;\wedge\; \frac{a(n,k+1)}{a(n,k)} = r_2(n,k).$$

Example 79. *The summand above, $a(n,k) = \binom{n}{k}$ is of this type, as*

$$r_1(n,k) = \frac{\binom{n+1}{k}}{\binom{n}{k}} = \frac{(n+1)!}{k!(n+1-k)!}\frac{k!(n-k)!}{n!} = \frac{n+1}{n+1-k},$$

and

$$r_2(n,k) = \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n!}{(k+1)!(n-k-1)!}\frac{k!(n-k)!}{n!} = \frac{n-k}{k+1}.$$

Assume that

$$s(n) = \sum_{k=0}^{n} a(n,k)$$

satisfies a holonomic recurrence of order one (i.e., is hypergeometric). Then there exist polynomials c_1, c_2 s.t.,

$$c_0(n)s(n) + c_1(n)s(n+1) = 0.$$

Plugging in the definition of $s(n)$ yields,

$$c_0(n)\sum_{k=0}^{n} a(n,k) + c_1(n)\sum_{k=0}^{n+1} a(n+1,k) = 0$$

$$\sum_{k=0}^{n} \underbrace{[c_0(n)a(n,k) + c_1(n)a(n+1,k)]}_{[c_0(n)+c_1(n)r_1(n,k)]a(n,k) =: \tilde{a}(n,k)} = -c_1(n)a(n+1,n+1).$$

Since $a(n,k)$ is hypergeometric in both n and k, we have that $a(n+1,n+1)$ is hypergeometric and that $\tilde{a}(n,k)$ is also hypergeometric both in n and k.

Zeilberger's idea was to apply Gosper's algorithm to $\tilde{a}(n,k)$ with c_0, c_1 as additional variables and fix n.

Gosper cannot return a hypergeometric solution for any c_0, c_1, but it can be modified to determine c_0, c_1 such that a hypergeometric solution exists (if there is one). This method is also sometimes referred to as *parametrized Gosper*.

1. Compute the shift quotient:
$$\frac{\tilde{a}(n,k+1)}{\tilde{a}(n,k)} = \frac{c_0(n)+c_1(n)r_1(n,k+1)}{c_0(n)+c_1(n)r_1(n,k)} \underbrace{\frac{a(n,k+1)}{a(n,k)}}_{=r_2(n,k)},$$
and define
$$\tilde{r}_2(z,x) = \frac{c_0+c_1r_1(z,x+1)}{c_0+c_1r_1(z,x)} r_2(z,x) \in \mathbb{K}(c_0,c_1,z)(x).$$
Example 80. *(Ex. 79 continued) We had* $a(n,k)=\binom{n}{k}$, $r_1(n,k)=\frac{n+1}{n+1-k}$, $r_2(n,k)=\frac{n-k}{k+1}$. *Hence,*
$$\tilde{r}_2(z,x) = \frac{(z-x)c_0+(z+1)c_1}{(z-x+1)c_0+(z+1)c_1}\frac{z-x+1}{x+1}$$

2. Determine the *Gosper form* of $\tilde{r}_2(z,x)$:
$$\tilde{r}_2(z,x) = \frac{t(z,x+1)}{t(z,x)}\frac{u(z,x)}{v(z,x+1)}, \quad \text{with} \quad \gcd(u(z,x),v(z,x+h))=1, \quad h\geq 1.$$
Example 81. *(Ex. 79 continued) In our running example, we can choose*
$$t(z,x)=(z-x+1)c_0+(z+1)c_1, \quad u(z,x)=z-x+1, \quad v(z,x)=x.$$

3. Set up the *Gosper equation*:
$$u(z,x)y(z,x+1)-v(z,x)y(z,x)=t(z,x).$$
Example 82. *(Ex. 79 continued) We have,*
$$(z-x+1)y(z,x+1)-xy(z,x)=(z-x+1)c_0+(z+1)c_1.$$
Note that the left hand side is independent of the constants c_i and that in the right hand side the c_i appear linearly.

4. (Following Gosper's idea:) Solve for $y(z,x)$ *polynomial* in x. First, we fix the degree d (using the degree bound we derived in the Algorith POLY). (Note, that the c_i do not depend on x.) Next, set up an ansatz
$$y(z,x)=\sum_{j=0}^{d} y_j(z)x^j \in \mathbb{K}(z)[x],$$
plug into the Gosper equation and compare like powers of x to solve for $y_0,\ldots,y_d,c_0,c_1$ simultaneously.

Example 83. *(Ex. 79 continued) Try $d = 0$ (coming from POLY), i.e., $y(z, x) = y_0(z)$:*

$$(z - x + 1)y_0 - xy_0 = (z - x + 1)c_0 + (z + 1)c_1$$
$$-2y_0 \cdot x + (z + 1)y_0 = -c_0 \cdot x + (z + 1)(c_0 + c_1).$$

In this case, we find the non-trivial solution $y_0 = \frac{c_0}{2}, c_1 = -\frac{c_0}{2}$.

5. If there is a non-trivial solution (i.e., not both $c_0, c_1 = 0$), plug back in

$$w(z, x) = \frac{v(z, x)}{t(z, x)} y(z, x)$$

to obtain $b_k = w_k a_k$ with $a_k = b_{k+1} - b_k$.

Example 84. *(Ex. 79 continued) Plugging in yields*

$$w(z, x) = \frac{x}{(z - x + 1)c_0 + (z + 1)c_1} \frac{c_0}{2} = \frac{x}{(z - x + 1)c_0 - (z + 1)c_0/2} \frac{c_0}{2} = \frac{x}{z - 2x + 1}.$$

6. Plug the solution into the telescoping equation

$$c_0(n)a(n, k) + c_1(n)a(n + 1, k) = \Delta_k [w(n, k)a(n, k)].$$

Since the coefficients on the left hand side are independent of k, we can sum both sides over k. By telescoping we obtain a recurrence for the sum (possibly with a hypergeometric term on the right hand side).

Example 85. *(Ex. 79 continued) Summarizing, we have*

$$c_0 a(n, k) - \frac{c_0}{2} a(n + 1, k) = \Delta_k \left[\frac{k}{n - 2k + 1} a(n, k) \right],$$

with $a(n, k) = \binom{n}{k}$ and $s(n) = \sum_{k=0}^{\infty} \binom{n}{k}$ (note that this sum has natural boundaries!). Summing both sides of the telescoping equation over $k = 0, \ldots, \infty$ gives,

$$c_0 s(n) - \frac{c_0}{2} s(n + 1) = \sum_{k=0}^{\infty} \Delta_k \left[\frac{k}{n - 2k + 1} a(n, k) \right].$$

For the right hand side we note that

$$k = 0 : \quad \frac{0}{n + 1} \binom{n}{0} = 0 \quad \textit{and} \quad \lim_{k \to \infty} k + 1n - 2k + 1 \binom{n}{k} = 0, \qquad n \geq 0.$$

Hence, we end up with the recurrence relation,

$$s(n) - \frac{1}{2} s(n + 1) = 0, \qquad n \geq 0, \quad s(0) = 1,$$

which can easily be solved to find $s(n) = 2^n$.

7. If in Step 4 no solution was found, increase the order

$$c_0(n)a(n,k) + c_1(n)a(n+1,k) + c_2(n)a(n+2,k) = \tilde{a}(n,k)$$

and apply Gosper's algorithm (goto Step 1).

The following definition and theorem state for which type of summand this process terminates.

Definition 86. *A term $a(n,k)$ is called* proper hypergeometric *if it can be written as*

$$a(n,k) = p(n,k)\frac{\prod_i'(a_i n + b_i k + c_i)!}{\prod_j'(\alpha_j n + \beta_j k + \gamma_j)!} z^k,$$

with

- $p \in \mathbb{K}[z,x]$,
- $a_i, b_i, c_i, \alpha_i, \beta_i, \gamma_i$ *are specific integers,*
- $z \in \mathbb{C}$ *or z an indeterminate,*

and where $\prod'$ denotes a finite product (of fixed length).

Theorem 87. *Let $(a(n,k))_{n,k\geq 0}$ be a proper hypergeometric sequence. Then there exists an order $r \in \mathbb{N}$ and polynomials $c_0, \ldots, c_r \in \mathbb{K}[z]$ not all zero, and $w(z,x) \in \mathbb{K}(z,x)$ s.t.,*

$$c_0(n)a(n,k) + c_1(n)a(n+1,k) + \cdots + c_r(n)a(n+r,k) = \Delta_k \left[w(n,k)a(n,k)\right].$$

For further details see, e.g., the book "A=B" [4].

8 Hypergeometric solutions of holonomic recurrences

Next, we discuss an algorithm to find all hypergeometric solutions to a holonomic recurrence. It is due to *Marko Petkovšek* and reduces the problem of finding hypergeometric solutions to finding polynomial solutions - which is something we know how to do. Similar to the derivation of the Algorith POLY, we only discuss the case order $r = 2$. The general case extends analogously. Summarizing, we are looking at the following problem:
Given $a_0, a_1, a_2 \in \mathbb{K}[x]$

Find all *hypergeometric* solutions of

$$a_2(n)y(n+2) + a_1(n)y(n+1) + a_0(n)y(n) = 0. \tag{24}$$

Assume $y(n)$ is a hypergeometric solution of (24), then

$$\exists s \in \mathbb{K}(x)\colon y(n+1) = s(n)y(n).$$

If we plug this into the recurrence, we obtain

$$a_2(n)s(n+1)s(n)y(n) + a_1(n)s(n)y(n) + a_0(n)y(n) = 0,$$

i.e., assuming we have shifted beyond all zeroes of $y(n)$, we are looking for a rational solution s to

$$a_2(n)s(n+1)s(n) + a_1(n)s(n) + a_0(n) = 0. \tag{25}$$

Let's write the (unknown) rational solution $s(n)$ in its *Gosper-Petkovšek* form:

$$s(n) = \frac{t(n+1)}{t(n)}\frac{u(n)}{v(n+1)}z,$$

with

- $t, u, v \in \mathbb{K}[x]$ *monic*, $z \in \mathbb{K} \setminus \{0\}$,
- $\gcd(u(x), v(x+h)) = 1$, $h \geq 1$, and
- $\gcd(u(x), t(x)) = 1$ and $\gcd(t(x+1), v(x+1)) = 1$.

This generalization of the Gosper form was observed by Marko Petkovšek and it makes the representation unique. Now, we plug this into (25),

$$a_2(n)\frac{t(n+2)}{t(n+1)}\frac{u(n+1)}{v(n+2)}\frac{t(n+1)}{t(n)}\frac{u(n)}{v(n+1)}z^2 + a_1(n)\frac{t(n+1)}{t(n)}\frac{u(n)}{v(n+1)}z + a_0(n) = 0.$$

After cancelation and clearing denominators, we have

$$a_2(n)t(n+2)u(n+1)u(n)z^2 + a_1(n)v(n+2)t(n+1)u(n)z + a_0(n)v(n+2)v(n+1)t(n) = 0.$$

From this, using the properties of the Gosper-Petkovšek form, we can derive the following two conditions:

$$\begin{aligned} u(n)\big|a_0(n)v(n+2)v(n+1)t(n) \quad &\Longrightarrow \quad u(n)\big|a_0(n), \\ v(n+2)\big|a_2(n)t(n+2)u(n+1)u(n) \quad &\Longrightarrow \quad v(n+2)\big|a_2(n). \end{aligned}$$

This gives a *finite* set of candidates for u and v: all possible combinations of monic, irreducible factors of $a_0(n)$ and $a_2(n-2)$. Furthermore, we can cancel $u(n)$ and $v(n+2)$ in (25),

$$z^2\frac{a_2(n)}{v(n+2)}u(n+1)t(n+2) + za_1(n)t(n+1) + \frac{a_0(n)}{u(n)}v(n+1)t(n) = 0. \tag{26}$$

Note, that this is an equation with all polynomial coefficients. Every choice of u, v gives rise to a quadratic equation for z (by equating the leading coefficient in n to zero) with at most 2 solutions: For every choice of u, v, z, apply POLY to (26).

Example 88. *This is a continuation of the Example 66 for Algorithm POLY. We consider the following recurrence,*

$$(n-1)y(n+2) - (3n-2)y(n+1) + 2ny(n) = 0, \quad y(0) = 1,\ y(1) = 3.$$

We are looking for a hypergeometric solution with

$$y(n+1) = s(n)y(n) \quad \text{and} \quad s(n) = \frac{t(n+1)}{t(n)}\frac{u(n)}{v(n+1)}z.$$

Now we determine the set of candidates for u, v:

$$\begin{aligned} v(n+2)\big|a_2(n) = n-1 \quad &\Longrightarrow v(n+2) = 1, n-1 \\ u(n)\big|a_0(n) = 2n \quad &\Longrightarrow u(n) = 1, n. \end{aligned}$$

To find all hypergeometric solutions (if there are any), we need to possibly check all combinations of these candidates, i.e., in this case $(u, v) \in \{(1,1), (1, n-3), (n, 1), (n, n-3)\}$. In our first try, we plug $v(n+2) = u(n) = 1$ into (26),

$$(n-1)z^2 - (3n-2)z + 2n = 0.$$

From the leading coefficient w.r.t. n we get the equation

$$z^2 - 3z + 2 = (z-1)(z-2) = 0,$$

and thus we have the two candidates $z_1 = 1, z_2 = 2$ for z. If we try POLY on the combination $(u, v, z) = (1, 1, 1)$, we arrive at the equation

$$(n-1)t(n+2) - (3n-2)t(n+1) + 2nt(n) = 0,$$

for which we already determined the general solution $t_0(n) = c_0 n$. Trying POLY on the combination $(u, v, z) = (1, 1, 2)$, i.e., on

$$4(n-1)t(n+2) - 2(3n-2)t(n+1) + 2nt(n) = 0,$$

we obtain the solution $t_1(n) = c_1$. Since we are considering a second order recurrence and have now found two linear independent solutions, we are done and we can compute the hypergeometric solutions:

$$s_0(n) = \frac{n+1}{n} \quad \Rightarrow \quad y_0(n+1) = \frac{n+1}{n} y(n) \quad \Rightarrow \quad y_0(n) = C_0 n.$$

and

$$s_1(n) = 2\frac{c_1}{c_1}\frac{1}{1} \quad \Rightarrow \quad y_1(n+1) = 2y_1(n) \quad \Rightarrow \quad y_1(n) = C_1 2^n.$$

Hence the general solution is $y(n) = C_0 n + C_1 2^n$ and the constants C_i can be computed from the initial values.

Marko Petkovšek wrote a Mathematica implementation of the algorithm under the name *Hyper* and it is available at

`https://www.fmf.uni-lj.si/~petkovsek/software.html`

An application of Hyper is the factorization of difference operators as it computes first order right factors. As before, let S_n denote the forward shift operator acting on n. Let $p \in \mathbb{K}(x)$, then

$$S_n p(n) = p(n+1) S_n.$$

The right hand side can be viewed as operator multiplication of a zero order operator $(p(n+1))$ with a first order operator (S_n). To divide operators from the *right*, we use the formula,

$$p(n) S_n^k = \left(\frac{p(n)}{q(n+k-m)} S_n^{k-m} \right) (q(n) S_n^m),$$

for some $q \in \mathbb{K}(x)$. If we know how to divide monomials, we can divide operators.

Let $L_1, L_2 \in \mathbb{K}(n)[S_n]$, $L_2 \neq 0$, then there are operators $Q, R \in \mathbb{K}(n)[S_n]$ s.t.

$$L_1 = QL_2 + R \quad \text{with} \quad \operatorname{ord}(R) < \operatorname{ord}(L_2),$$

where ord denotes the order of an operator. So we have right division with remainder. Furthermore, we have a *right Euclidean domain* and can compute *greatest common right divisors* (gcrd) and *least common left multiples* (lclm).

Say we are given $L_1, L_2 \in \mathbb{K}(n)[S_n]$ with $L_1 y_1 = 0$ and $L_2 y_2 = 0$ for sequences $y_1, y_2 \in \mathbb{K}^{\mathbb{N}}$. Now let $M = \operatorname{lclm}(L_1, L_2)$, i.e., there exist operators $M_1, M_2 \in \mathbb{K}(n)[S_n]$ s.t.,

$$M = M_1 L_1 = M_2 L_2$$

and

$$M(y_1 + y_2) = M_1 L_1 y_1 + M_2 L_2 y_2 = M_1 0 + M_2 0 = 0.$$

Given a sequence $a \in \mathbb{K}^{\mathbb{N}}$ and an annihilating operator $L \in \mathbb{K}(n)[S_n]$ with $La = 0$. Then there exists a *minimal order* operator M with $Ma = 0$. and we can right divide L by M: Let

$$L = QM + R \quad \text{with} \quad \operatorname{ord}(R) < \operatorname{ord}(M),$$

and we have

$$\underbrace{La}_{=0} = Q \underbrace{Ma}_{=0} + Ra \quad \Longrightarrow \quad Ra = 0 \quad \Longrightarrow R = 0,$$

where in the last step we used the minimality of M.

Hence, the minimal order annihilating operator of a is a right factor of any annihilating operator of a.

Hypergeometric solutions of $Ly = 0$ correspond to monic first order right factors of L.

Example 89. *For the recurrence*

$$(n-1)y(n+2) - (3n-2)y(n+1) + 2ny(n) = 0$$

we have found the two hypergeometric solutions

$$y_0(n) = C_0 n, \quad \textit{with recurrence} \quad y_0(n+1) = \frac{n+1}{n} y_0(n),$$

and

$$y_1(n) = C_1 2^n \quad \textit{with recurrence} \quad y_1(n+1) = 2y_1(n).$$

Let's write the given second order recurrence in operator form

$$(n-1)S_n^2 - (3n-2)S_n + 2n$$

and right divide the two operators corresponding to the two hypergeometric solutions. First, we use the simpler operator $S_n - 2$ corresponding to y_1:

$$\begin{array}{l} \big((n-1)S_n^2 - (3n-2)S_n + 2n\big) : (S_n - 2) = (n-1)S_n - n \\ \underline{(n-1)S_n^2 - 2(n-1)S_n \qquad\qquad} \\ \qquad\qquad\qquad\quad -nS_n + 2n \\ \underline{\qquad\qquad\qquad\quad -nS_n + 2n} \\ \qquad\qquad\qquad\qquad\qquad\quad 0 \end{array}$$

So, we found that we can factor L as

$$L = ((n-1)S_n - n)\,(S_n - 2).$$

On the other hand, if we right divide by the operator $S_n - \frac{n+1}{n}$*, we obtain*

$$
\begin{array}{l}
\big((n-1)S_n^2 - (3n-2)S_n + 2n\big) : \left(S_n - \dfrac{n+1}{n}\right) = (n-1)S_n - \dfrac{2n^2}{n+1} \\
\underline{(n-1)S_n^2 + (n-1)\left(-\dfrac{n+2}{n+1}\right)S_n \quad} \\
\qquad\qquad -\dfrac{2n^2}{n+1}S_n + 2n \\
\underline{\qquad\qquad -\dfrac{2n^2}{n+1}S_n + \dfrac{2n^2}{n+1}\dfrac{n+1}{n}} \\
\qquad\qquad\qquad\qquad 0
\end{array}
$$

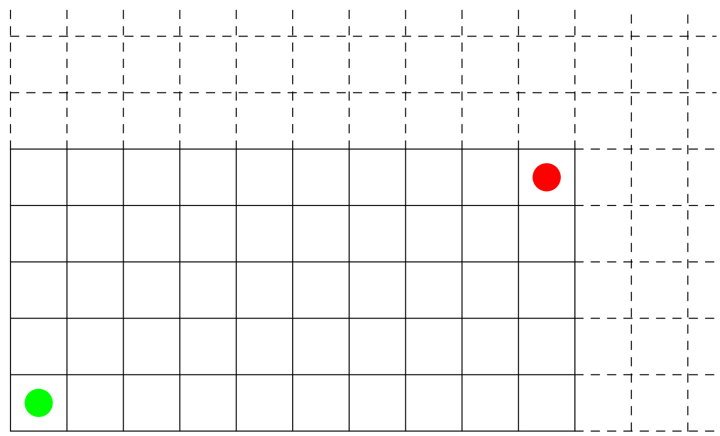
yielding

$$L = \left((n-1)S_n - \frac{2n^2}{n+1}\right)\left(S_n - \frac{n+1}{n}\right).$$

9 A bivariate example: rook walks

Let us assume that we are moving a rook on an infinite dimensional chess board (the first quadrant) from the origin $(0,0)$ to some endpoint (i,j) on the board. The *possible moves* of a rook are

- direction: going North $\uparrow$ and going East $\rightarrow$,
- length: arbitrary,
- there is *NO* going back.



We want to determine

$$a(i,j) = \#(\text{paths starting at } (0,0) \text{ and ending at } (i,j)).$$

There is no simple closed form for this sequence, but we can determine a recurrence: $a(i,j)$ equals the number of possible positions before the last move, i.e.,

$$a(i,j) = \sum_{k=0}^{j-1} a(i,k) + \sum_{k=0}^{i-1} a(k,j), \qquad (i,j) \in \mathbb{N}^2 \setminus \{(0,0)\}, \quad a(0,0) = 1. \tag{27}$$

Next, let

$$\alpha(n) \coloneqq a(n,n) = \#(\text{number of paths on an } n \times n \text{ chess board from the lower left to the upper right corner}).$$

The cost for computing $\alpha(n)$ using the recurrence (27) is $\mathcal{O}(n^3)$. Could we do better than that?

Let's have a look at the bivariate generating function

$$\begin{aligned}
f(x,y) &= \sum_{i,j\geq 0} a(i,j)x^i y^j \\
&= 1 + \sum_{(i,j)\in\mathbb{N}^2\setminus\{(0,0)\}} a(i,j)x^i y^j \\
&= 1 + \sum_{(i,j)\in\mathbb{N}^2\setminus\{(0,0)\}} \left(\sum_{k=0}^{j-1} a(i,k) + \sum_{k=0}^{i-1} a(k,j)\right) x^i y^j.
\end{aligned}$$

We start with the first sum and fix i: note, that

$$\sum_{k=0}^{j-1} a(i,k) = 0 \quad \text{for} \quad j = 0,$$

and hence for all $i \geq 0$ we must have $j \geq 1$. Then for $i \geq 0$,

$$\begin{aligned}
\sum_{j\geq 1}\sum_{k=0}^{j-1} a(i,k)y^j &= \sum_{j\geq 0}\sum_{k=0}^{j} a(i,k)y^{j+1} \\
&= y\left(\sum_{j\geq 0} a(i,j)y^j\right)\left(\sum_{j\geq 0} y^j\right) \\
&= \frac{y}{1-y}\sum_{j\geq 0} a(i,j)y^j.
\end{aligned}$$

In the same way, we can argue that for fixed $j \geq 0$ we have

$$\sum_{i\geq 1}\sum_{k=0}^{i-1} a(k,j)x^i = \cdots = \frac{x}{1-x}\sum_{i\geq 0} a(i,j)x^i.$$

Summarizing, we obtain the following functional equation for $f(x,y)$,

$$f(x,y) = 1 + \frac{y}{1-y}f(x,y) + \frac{x}{1-x}f(x,y).$$

From this, we can easily compute the closed form

$$f(x,y) = \frac{(1-x)(1-y)}{1-2x-2y+3xy},$$

which in turn allows to derive a C-finite recurrence for $a(i,j)$. First, let

$$g(x,y) = \frac{1}{1-2x-2y+3xy} = \sum_{i,j\geq 0} b(i,j)x^i y^j.$$

Then

$$\begin{aligned}
1 &= (1-2x-2y+3xy)\sum_{i,j\geq 0} b(i,j)x^iy^j \\
&= \sum_{i,j\geq 0}\big(b(i,j)x^iy^j - 2b(i,j)x^{i+1}y^j - 2b(i,j)x^iy^{j+1} + 3b(i,j)x^{i+1}y^{j+1}\big) \\
&= \sum_{i,j\geq 0} b(i,j)x^iy^j - \sum_{i\geq 1, j\geq 0} 2b(i-1,j)x^iy^j - \sum_{i\geq 0, j\geq 1} 2b(i,j-1)x^iy^j \\
&\qquad + \sum_{i,j\geq 1} 3b(i-1,j-1)x^iy^j.
\end{aligned}$$

Coefficient comparison for $i, j \geq 1$ gives the recurrence

$$[x^iy^j]: \quad 0 = b(i,j) - 2b(i-1,j) - 2b(i,j-1) + 3b(i-1,j-1),$$

and the initial values

$$\begin{aligned}
&[x^0y^0]: \quad 1 = b(0,0), \\
&[x^0y^j]: \quad 0 = b(0,j) - 2b(0,j-1) \quad \Rightarrow \quad b(0,j) = 2^j, \quad j \geq 1, \\
&[x^iy^0]: \quad 0 = b(i,0) - 2b(i-1,0) \quad \Rightarrow \quad b(i,0) = 2^i, \quad i \geq 1.
\end{aligned}$$

Hence, summarizing, we have

$$\begin{aligned}
&b(i+1,j+1) = 2b(i,j+1) + 2b(i+1,j) - 3b(i,j), \quad i,j \geq 0, \\
&b(0,k) = b(k,0) = 2^k, \quad k \geq 0.
\end{aligned}$$

In operator notation we thus can write $Lb = 0$ for

$$L = S_iS_j - 2S_i - 2S_j + 3.$$

Note, that

$$\begin{aligned}
(1-x)\sum_{n\geq 0} c_nx^n &= \sum_{n\geq 0} c_nx^n - \sum_{n\geq 1} c_{n-1}x^n \\
&= c_0 + \sum_{n\geq 1}(c_n - c_{n-1})x^n \\
&= c_0 + \sum_{n\geq 0}\underbrace{(c_{n+1} - c_n)}_{=(S_n-1)c(n)}x^{n+1}
\end{aligned}$$

Since $f(x,y) = (1-x)(1-y)g(x,y)$, we thus have

$$a(i,j) = (S_i-1)(S_j-1)b(i,j), \qquad i,j \geq 1. \tag{28}$$

What happens if we apply this operator L to $a(i,j)$?

$$La = L(S_i-1)(S_j-1)b = (S_i-1)(S_j-1)Lb = 0.$$

The operators commute, because their coefficients are constant, and hence L also annihilates a. Since (28) only holds for $i, j \geq 1$, we have to provide sufficiently many initial values. Consequently, we have

$$a(i+1, j+1) = 2a(i, j+1) + 2a(i+1, j) - 3a(i, j), \qquad i, j \geq 1, \tag{29}$$

with

$$a(0,0) = 1, \quad a(0,k) = a(k,0) = 2^{k-1}, \quad a(1,k) = a(k,1) = 2^{k-2}(k+3), \quad k \geq 1.$$

Using the recurrence (29), the complexity for computing $\alpha(n)$ goes down to $\mathcal{O}(n^2)$. There is even a faster way: using *Guessing* we find

$$(n+2)\alpha(n+2) - (10n+14)\alpha(n+1) + 9n\alpha(n) = 0, \qquad \alpha(0) = 1,\ \alpha(1) = 2, \quad n \geq 0. \tag{30}$$

With this, the complexity is linear ($\mathcal{O}(n)$).

Remark 90. *(Lipshitz, 1988) If $\sum_{i,j} a(i,j) x^i y^j$ is holonomic, then $\sum_{n \geq 0} a(n,n) z^n$ is also holonomic. This results extends to more variables than two.*

10 Asymptotics of holonomic sequences

We briefly discuss what type of asymptotic behaviour holonomic sequences can have. First, recall that we defined two sequences a_n, b_n to be asymptotically equivalent as follows,

$$a_n \sim b_n (n \to \infty) \quad \Longleftrightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

For hypergeometric sequences, we noted that

$$\frac{(c_1)_n}{(c_2)_n} \sim \frac{\Gamma(c_1)}{\Gamma(c_2)} n^{c_1 - c_2} (n \to \infty),$$

where $(a)_n$ denotes the Pochhammer symbol (or rising factorial).

For holonomic sequences, finding the asymptotic behaviour is hard in general. If the generating function is analytic, it is possible to determine the type of asymptotics of a sequence.

Definition 91. *Let a, b be complex functions,* analytic *in a neighbourhood of $z_0 \in \mathbb{C}$. Then a, b are* asymptotically equivalent at z_0 *iff*

$$\lim_{z\to z_0} \frac{a(z)}{b(z)} = 1.$$

Notation: $a(z) \sim b(z)(z \to z_0)$.

Let $a(z) = \sum_{n\geq 0} a_n z^n$ be analytic in a neighbourhood of 0 containing 1. Then,

$$a(z) \sim (1-z)^{\alpha} \log(1-z)^{\beta}\,(z \to 1) \quad \Leftrightarrow \quad a_n \sim \frac{(-1)^{\beta}}{\Gamma(-\alpha)} n^{-\alpha-1} \log(n)^{\beta}\,(n \to \infty),$$

with $\alpha \notin \mathbb{N}, \beta \in \mathbb{N}$. These are the types of singularities that can be detected.

There are two obvious issues:

- the generating function is not analytic
- the coefficients grow too slow (there are no singularities)

In these two cases, sometimes scaling helps.

Example 92. *We found earlier that the generating function for average QuickSort counting is*

$$A(z) = -\frac{2z}{(1-z)^2} - \frac{2}{(1-z)^2} \log(1-z).$$

The second term is the dominating term for $z \to 1$ and we read off $\alpha = -2, \beta = 1$ and thus find

$$a(n) \sim 2\frac{(-1)^1}{\Gamma(2)} n^{2-1} \log(n)^1 = 2n \log(n)\,(n \to \infty).$$

References

[1] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[2] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994.

[3] M. Kauers and P. Paule. *The Concrete Tetrahedron: Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates*. Springer Publishing Company, Incorporated, 1st edition, 2011.

[4] M. Petkovšek, H.S. Wilf, and D. Zeilberger. $A = B$. A K Peters Ltd., Wellesley, MA, 1996.

[5] R.P. Stanley. *Enumerative Combinatorics*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2 edition, 2011.

[6] R.P. Stanley and S. Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.

[7] H.S. Wilf. *Generatingfunctionology*. Academic Press, 1990.