In[1]:= `<< RISC`fastZeil``

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese

In[2]:= `? Gosper`

Gosper[function, range],
uses Gosper's algorithm to find a hypergeometric closed form for the sum
of the function over the range,

Gosper[function, k], computes the hypergeometric forward anti–difference
of function in k, if it exists,

Gosper[function, range, degree] or

Gosper[function, k, degree] use Gosper's algorithm with an
undetermined polynomial of given degree in k multiplied to the function.

Example 78:

In[3]:= `Gosper[k k!, k]`

Out[3]= $\{k\,k! == \Delta_k[k!]\}$

In[4]:= `Gosper[k k!, {k, 0, n}]`

```
If `n' is a natural number, then:
```

Out[4]= `{Sum[k k!, {k, 0, n}] == -1 + (1 + n) n!}`

The factorial does not have a hypergeometric antidifference:

In[5]:= `Gosper[k!, k]`

Out[5]= `{ }`

The harmonic numbers are not hypergeometric:

In[6]:= `Gosper[1/k, {k, 1, n}]`

Out[6]= `{ }`

Homework example:

In[7]:= `Gosper[Binomial[2 k, k] α^k, {k, 0, n}]`

Out[7]= `{ }`

In[8]:=
```
Gosper[Binomial[2 k, k] (1 / 4) ^k, {k, 0, n}]
```

If `n' is a natural number, then:

Out[8]= $\{\text{Sum}[4^{-k}\,\text{Binomial}[2\,k, k], \{k, 0, n\}] == 4^{-n}\,(1 + 2\,n)\,\text{Binomial}[2\,n, n]\}$

In[9]:=
```
Gosper[Binomial[x + k, k], {k, 0, n}]
```

If `n' is a natural number and $1 + x \neq 0$, then:

Out[9]= $\left\{\text{Sum}[\text{Binomial}[k + x, k], \{k, 0, n\}] == \frac{(1 + n + x)\,\text{Binomial}[n + x, n]}{1 + x}\right\}$

In[10]:=
```
Gosper[Binomial[n, k], {k, 0, n}]
```

Out[10]= { }

In[11]:=
```
? Zb
```

Zb[function, range, n, order],
uses Zeilberger's algorithm to find a recurrence relation of given order in n
for the sum of the function over the range.

Zb[function, k, n, order],
uses Zeilberger's algorithm to find a recurrence relation of given order in n
for the function. This recurrence is — up to a telescoping part —
free of k.

In both calls, if the order is of the form {ord1, ord2}, Zb tries to find
a recurrence whose order is between ord1 and ord2. Omitting the order is equivalent to
specifying {0, Infinity}.

In[12]:=
```
Zb[Binomial[n, k], {k, 0, n}, n]
```

If `n' is a natural number, then:

Out[12]= {2 SUM[n] - SUM[1 + n] == 0}

In[13]:=
```
Zb[Binomial[n, k], {k, 0, Infinity}, n]
```

Out[13]= {2 SUM[n] - SUM[1 + n] == 0}

In[14]:=
```
Zb[Binomial[n, k], k, n]
```

Out[14]= $\{2\,F[k, n] - F[k, 1 + n] == \Delta_k[F[k, n]\,R[k, n]]\}$

In[15]:=
```
Prove[]
```

In[16]:=
```
Zb[Binomial[n, k]^2, {k, 0, Infinity}, n]
```

Out[16]= {-2 (1 + 2 n) SUM[n] + (1 + n) SUM[1 + n] == 0}

In[17]:=
```
Zb[(-1)^k Binomial[n, k] / Binomial[x + k, k], k, n]
```

Out[17]=
$$\left\{\frac{(-1)^k \text{Binomial}[n, k]}{\text{Binomial}[k + x, k]} == \Delta_k\left[\frac{(-1)^k (k + x) \text{Binomial}[n, k]}{(-n - x) \text{Binomial}[k + x, k]}\right]\right\}$$

Jacobi polynomials

In[18]:=
```
Zb[Pochhammer[-n, k]
  Pochhammer[n + α + β + 1, k] / Pochhammer[α + 1, k] / k! ((1 - x) / 2)^k, {k, 0, n}, n]
```

If `n' is a natural number, then:

Out[18]=
$$\{-2 (1 + n) (1 + n + \beta) (4 + 2 n + \alpha + \beta) \text{SUM}[n] + (3 + 2 n + \alpha + \beta) (8 x + 12 n x + 4 n^2 x + 6 x \alpha + 4 n x \alpha + \alpha^2 + x \alpha^2 + 6 x \beta + 4 n x \beta + 2 x \alpha \beta - \beta^2 + x \beta^2) \text{SUM}[1 + n] - 2 (2 + n + \alpha) (2 + n + \alpha + \beta) (2 + 2 n + \alpha + \beta) \text{SUM}[2 + n] == 0\}$$

Non-minimality of Zeilberger's algorithm:

In[19]:=
```
d = 1;
Zb[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}, n]
```

If `n' is a natural number, then:

Out[20]=
$$\left\{\text{SUM}[n] + \text{SUM}[1 + n] == -\frac{n \text{Binomial}[0, n]}{1 + n}\right\}$$

In[21]:=
```
d = 2;
Zb[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}, n]
```

If `n' is a natural number, then:

Out[22]=
$$\{-2 (1 + n) \text{SUM}[n] + (-1 - n) \text{SUM}[1 + n] == -n \text{Binomial}[0, n]\}$$

In[23]:=
```
d = 3;
rec = Zb[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}, n]
```

If `n' is a natural number, then:

Out[24]=
$$\{9 (1 + n) (2 + n) \text{SUM}[n] + 3 (2 + n) (7 + 5 n) \text{SUM}[1 + n] + 2 (2 + n) (3 + 2 n) \text{SUM}[2 + n] == -n (3 + 2 n) \text{Binomial}[0, n]\}$$

In[25]:=
```
FullForm[rec]
```

Out[25]//FullForm=
```
List[Equal[Plus[Times[9, Plus[1, n], Plus[2, n], HoldForm[SUM[n]]],
    Times[3, Plus[2, n], Plus[7, Times[5, n]], HoldForm[SUM[Plus[1, n]]]],
    Times[2, Plus[2, n], Plus[3, Times[2, n]], HoldForm[SUM[Plus[2, n]]]]],
  Times[-1, n, Plus[3, Times[2, n]], Binomial[0, n]]]]
```

In[26]:=
```
rec = ReleaseHold[rec[[1]]]
```

Out[26]=
$$9 (1 + n) (2 + n) \text{SUM}[n] + 3 (2 + n) (7 + 5 n) \text{SUM}[1 + n] + 2 (2 + n) (3 + 2 n) \text{SUM}[2 + n] == -n (3 + 2 n) \text{Binomial}[0, n]$$

In[27]:=
```
a[n_] := (-d)^n
```

In[28]:=
```
rec1 = rec /. SUM → a
```

Out[28]= $(-1)^n 3^{2+n} (1+n) (2+n) + 2 (-3)^{2+n} (2+n) (3+2n) + (-1)^{1+n} 3^{2+n} (2+n) (7+5n) ==$
$-n (3+2n)$ Binomial[0, n]

In[29]:=
```
FullSimplify[rec1]
```

Out[29]= $n (3+2n)$ Binomial[0, n] == 0

In[30]:=
```
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

Package GeneratingFunctions version 0.8 written by Christian Mallinger
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Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[31]:=
```
data = Table[Sum[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}], {n, 0, 50}];
```

In[32]:=
```
GuessMinRE[data, s[n]]
```

Out[32]= 3 s[n] + s[1 + n]