Summer Semester 2023

Symbolic Linear Algebra (selected slides)

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Lecture 1: March 7, 2023

Definition. Let $(\mathbb{G},+,\cdot)$ be a field and let $M\neq\emptyset$ be a set with two operations $+:M\times M\to M$ and $*:\mathbb{G}\times M\to M$.
$(M,+,*)$ is called a vector space over $\mathbb{G}$ (or a $\mathbb{G}$-vector space) if $(M,+)$ is an abelian group and in addition the following properties hold:

1. $\forall a\in M\ \ \forall\lambda,\mu\in\mathbb{G}:\ (\lambda\cdot\mu)*a=\lambda*(\mu*a)$;
2. $\forall a\in M:\ 1*a=a$ (here 1 is the neutral element in $\mathbb{G}$ w.r.t. $\cdot$);
3. $\forall a,b\in M\ \ \forall\lambda\in\mathbb{G}:\ \lambda*(a+b)=\lambda*a+\lambda*b$;
4. $\forall a\in M\ \forall\lambda,\mu\in\mathbb{G}:\ (\lambda+\mu)*a=\lambda*a+\mu*a$.

$*$ is also called a *scalar multiplication*.

Definition. Let $(\mathbb{G}, +, \cdot)$ be a ring and let $M \neq \emptyset$ be a set with two operations $+ : M \times M \to M$ and $* : \mathbb{G} \times M \to M$.
$(M, +, *)$ is called a left module over $\mathbb{G}$ (or a left $\mathbb{G}$-module) if $(M, +)$ is an abelian group and in addition the following properties hold:

1. $\forall a \in M \ \ \forall \lambda, \mu \in \mathbb{G} : \ (\lambda \cdot \mu) * a = \lambda * (\mu * a)$;
2. $\forall a \in M \ : \ 1 * a = a$ (here 1 is the neutral element in $\mathbb{G}$ w.r.t. $\cdot$);
3. $\forall a, b \in M \ \ \forall \lambda \in \mathbb{G} : \ \lambda * (a + b) = \lambda * a + \lambda * b$;
4. $\forall a \in M \ \forall \lambda, \mu \in \mathbb{G} : \ (\lambda + \mu) * a = \lambda * a + \mu * a$.

$*$ is also called a *scalar multiplication*.

Definition. Let $(\mathbb{G}, +, \cdot)$ be a ring and let $M \neq \emptyset$ be a set with two operations $+ : M \times M \to M$ and $* : \mathbb{G} \times M \to M$.
$(M, +, *)$ is called a right module over $\mathbb{G}$ (or a right $\mathbb{G}$-module) if $(M, +)$ is an abelian group and in addition the following properties hold:

1. $\forall a \in M \;\; \forall \lambda, \mu \in \mathbb{G} : \; a * (\mu \cdot \lambda) = (a * \mu) * \lambda$;
2. $\forall a \in M \; : \; a * 1 = a$ (here 1 is the neutral element in $\mathbb{G}$ w.r.t. $\cdot$);
3. $\forall a, b \in M \;\; \forall \lambda \in \mathbb{G} : \; (a + b) * \lambda = a * \lambda + b * \lambda$;
4. $\forall a \in M \; \forall \lambda, \mu \in \mathbb{G} : \; a * (\lambda + \mu) = a * \lambda + a * \mu$.

$*$ is also called a *scalar multiplication*.

Lecture 6: April 25, 2023

Theorem CHAR. Let R be a PID and $A \in M_n(R)$. Then the following statements are equivalent:

1. $A \in \mathsf{GL}_n(R)$
2. $\det(A) \in R^*$
3. $S_R(A) = R^n$.
4. The rows of A form a basis of R^n.
5. The columns of A form a basis of R^n.
6. A is row equivalent to I_n.
7. A is a product of elementary matrices.

Note: If R is commutative, the equivalences (1)–(5) hold.

Lecture 7: May 2, 2023

Lemma Q. Let R be a commutative ring, $A \in R^{m \times n}$, $b \in R^m$ and $Q \in \mathsf{GL}_n(R)$. Define

$$S_1 = \{x \in R^n \mid A\,x = b\},$$
$$S_2 = \{x \in R^n \mid A\,Q\,x = b\}.$$

Then: S_1 and S_2 are in $1-1$ correspondence with $f : S_1 \to S_2$ where $f(x) = Q^{-1}\,x$ and $f^{-1}(x) = Q\,x$.

Lemma P. Let R be a commutative ring, $A \in R^{m \times n}$, $b \in R^m$ and $P \in \mathsf{GL}_m(R)$. Define

$$S_1 = \{x \in R^n \mid A\,x = b\},$$
$$S_2 = \{x \in R^n \mid P\,A\,x = b\}.$$

Then: $S_1 = S_2$.