37. Let $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ be sequences. Prove that the difference operator satisfies the product rule
$$\Delta_n(f_n g_n) = (\Delta_n f_n)g_n + f_{n+1}(\Delta_n g_n), \qquad n \geq 0.$$
Use this law to deduce the formula for summation by parts,
$$\sum_{k=0}^{n} f_k(\Delta_k g_k) = f_{n+1}g_{n+1} - f_0 g_0 - \sum_{k=0}^{n}(\Delta_k f_k)g_{k+1}.$$

38. Suppose $a(x)$ is an algebraic formal power series and $b(x)$ is a formal power series with $a(b(x)) = x$. Show that $b(x)$ is also algebraic.

39. Let T_n be the number of tilings of a $3 \times n$ rectangle with straight trominoes (i.e., 1×3 and 3×1 pieces).
    (a) Determine a recurrence relation for T_n.
    (b) Express the partial sum $s_n = \sum_{k=0}^{n} T_k$ in terms of T_n.

40. Express $s_n = \sum_{k=0}^{n} a_k$ in terms of $a_n, a_{n+1}, \ldots$, where the sequence $\big(a_n\big)_{n\geq 0}$ is given by the recurrence
$$a_{n+3} = 5a_{n+1} - 4a_n, \qquad a_0 = a_1 = 1, \quad a_2 = 2.$$

The set of difference operators $\mathbb{K}[n]\langle S_n\rangle$ consists of elements $\sum_{i=0}^{r} p_i(n)S_n^i$ with $p_0, \ldots, p_r \in \mathbb{K}[n]$. We define the addition of two such operators elementwise (as for polynomials $\mathbb{K}[n][S_n]$) and the multiplication as
$$\left(\sum_{i=0}^{r} p_i(n)S_n^i\right)\left(\sum_{j=0}^{s} q_j(n)S_n^j\right) = \sum_{i=0}^{r+s}\sum_{j=0}^{i} p_j(n)q_{i-j}(n+j)S_n^i.$$
The set $\mathbb{K}[n]\langle S_n\rangle$ together with this addition and multiplication forms a non-commutative ring. Such an operator $A = \sum_{i=0}^{r} p_i(n)S_n^i$ can act on a sequence $a = (a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ as $A \cdot a = (\sum_{i=0}^{r} p_i(n)a_{n+i})_{n\geq 0}$.

41. Let $A, B \in \mathbb{K}[n]\langle S_n\rangle$ and $a \in \mathbb{K}^{\mathbb{N}}$. Show that
$$(AB)\cdot a = A \cdot (B \cdot a).$$

42. Show that a sequence $\big(a_n\big)_{n\geq 0}$ is holonomic if and only if there exist polynomials $p_0, \ldots, p_r \in \mathbb{K}[x]$ and a holonomic sequence $\big(b_n\big)_{n\geq 0}$ such that
$$p_r(n)a_{n+r} + \cdots + p_1(n)a_{n+1} + p_0(n)a_n = b_n, \qquad n \in \mathbb{N}.$$
Hint: You can use that a is holonomic if and only if there is a non-zero operator $A \in \mathbb{K}[n]\langle S_n\rangle$ with $A \cdot a = 0$.