

29. Determine the asymptotics of

$$a_n = \binom{2n+1}{n}^2 3^n (n+1),$$

i.e., find a simpler sequence b_n with $a_n \sim b_n (n \rightarrow \infty)$. Check on the computer how fast the quotient $\frac{a_n}{b_n}$ actually converges to 1.

30. The Catalan numbers C_n satisfy the linear recurrence

$$-(4n+2)C_n + (n+2)C_{n+1} = 0.$$

Derive a linear differential equation with polynomial coefficients for the generating function $f(x) = \sum_{n \geq 0} C_n x^n$.

31. Let $f(x) = \sin(x)$, $g(x) = e^{-x^2+x+1}$ be the holonomic functions satisfying the differential equations

$$f(x) + f''(x) = 0, \quad (1-2x)g(x) - g'(x) = 0.$$

Find a linear differential equation with polynomial coefficients for the function $h(x) = f(x) + g(x)$.

Hint: The same idea as in Example 44 can be used. Instead of considering shifts, you can consider derivatives and instead of solving a linear system over \mathbb{Q} , a linear system over $\mathbb{Q}(x)$ has to be solved.

32. Find a linear differential equation with polynomial coefficients for the holonomic functions $f'(x)$ and $\int_x f(x)$ where f is the function from Exercise 27, i.e., it satisfies

$$xf(x) + (x+1)f'(x) - f''(x) = 0.$$