17. Let the sequence $\big(f_n\big)_{n\geq 0}$ be recursively defined by

$$f_{n+3} - 2f_{n+2} - 5f_{n+1} + 6f_n = 0, \qquad f_0 = 0,\ f_1 = 8,\ f_2 = 2.$$

What is the companion matrix of this recurrence? Determine the solution of the recurrence analogously to the Fibonacci example presented in the lecture.

18. The easter bunny wants to arrange n eggs in a line. He has an infinite number of green, yellow and blue eggs, but two neighboring eggs in the line cannot be both green. Let e_n be the number of ways, the bunny can arrange n eggs.

    (a) Determine a recurrence for e_n.

    (b) Determine a closed form expression of e_n.

    (c) Determine the generating function of e_n.

19. Prove Theorem 40 for recurrences of order two, i.e., for $c_0 \neq 0, c_1 \in \mathbb{K}$ with

$$x^2 + c_1 x + c_0 = (x - \alpha_1)(x - \alpha_2)$$

with $\alpha_1 \neq \alpha_2 \in \mathbb{K}$, show that $(\alpha_1^n)_{n\geq 0}, (\alpha_2^n)_{n\geq 0}$ form a basis for the solutions of the recurrence

$$a_{n+2} + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

and that if $\alpha_1 = \alpha_2 = \alpha$ then $(\alpha^n)_{n\geq 0}, (n\alpha^n)_{n\geq 0}$ form a basis for the solutions of the above recurrence.

20. (Theorem 42) Show that a sequence $(a_n)_{n\geq 0}$ in $\mathbb{K}$ satisfies a C-finite recurrence

$$a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \qquad n \geq 0,$$

with $c_i \in \mathbb{K}, c_0 \neq 0$, if and only if

$$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0 x^r}$$

for some polynomial $p(x)$ with degree at most $r - 1$.