41. Find a nonzero polynomial $p \in \mathbb{K}[x]$ of minimal degree s.t.

$$\sum_{k=0}^{n} \frac{p(k)}{k!}$$

has a hypergeometric closed form.

42. Use Zeilberger's algorithm as presented in the lecture to determine a recurrence satisfied by $\sum_{k=0}^{n} \binom{n}{k} k$. You may use the information that the recurrence is of order one with linear coefficients.

43. Given are two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. The sequence a_n is hypergeometric satisfying $c_1(n)a_{n+1} = c_0(n)a_n$ for some polynomials c_0, c_1 and b_n is P-finite satisfying

$$d_r(n)b_{n+r} + \cdots + d_1(n)b_{n+1} + d_0(n)b_n = 0,$$

for polynomials d_k. Give a direct proof that the sequence $c_n = a_n b_n$ is P-finite of order r by explicitely computing the recurrence coefficients.

44. Prove an analog of the binomial theorem for factorial powers:

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$$

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}$$