

17. Let the sequence $(f(n))_{n \geq 0}$ be recursively defined by

$$f_{n+3} + 2f_{n+2} - f_{n+1} - 2f_n = 0, \quad f_0 = 1, f_1 = -1, f_2 = 2.$$

What is the companion matrix of this recurrence? Determine the solution of the recurrence analogously to the Fibonacci example presented in the lecture.

18. (Tower of Hanoi) Given a tower of n disks initially stacked in increasing order on one of three pegs, the task is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger disk onto a smaller one. Let a_n denote the minimal number of moves needed.

Find a recurrence for a_n . Compute the first few values and guess a closed form solution. Derive the closed form solution using techniques from the lecture.

19. Prove Theorem 40 for recurrences of order two, i.e., for $c_0 \neq 0, c_1 \in \mathbb{K}$ with

$$x^2 + c_1x + c_0 = (x - \alpha_1)(x - \alpha_2)$$

with $\alpha_1 \neq \alpha_2 \in \mathbb{K}$, show that $(\alpha_1^n)_{n \geq 0}, (\alpha_2^n)_{n \geq 0}$ form a basis for the solutions of the recurrence

$$a_{n+2} + c_1a_{n+1} + c_0a_n = 0, \quad n \geq 0,$$

and that if $\alpha_1 = \alpha_2 = \alpha$ then $(\alpha^n)_{n \geq 0}, (n\alpha^n)_{n \geq 0}$ form a basis for the solutions of the above recurrence.

20. (Theorem 42) Show that a sequence $(a_n)_{n \geq 0}$ in \mathbb{K} satisfies a C-finite recurrence

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1a_{n+1} + c_0a_n = 0, \quad n \geq 0,$$

with $c_i \in \mathbb{K}, c_0 \neq 0$, if and only if

$$\sum_{n \geq 0} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0x^r}$$

for some polynomial $p(x)$ with degree at most $r - 1$.