Recall the definition of *Stirling numbers of the second kind* $S_2(n,k)$ as the number of ways to partition an n-element set into a disjoint union of k nonempty subsets. They satisfy the recurrence relation

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k), \qquad n,k \geq 1,$$

with initial values $S_2(0,0) = 1$ and $S_2(n,0) = 0$ for $n \geq 1$, and $S_2(n,k) = 0$ for $k > n \geq 1$.

13. Show that for $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} S_2(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

14. Show that for $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} S_2(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\mathrm{e}^x - 1\right)^k,$$

and that

$$\sum_{n,k=0}^{\infty} S_2(n,k)\frac{x^n}{n!}y^k = \exp\left(y(\mathrm{e}^x-1)\right).$$

15. Let the signless Stirling numbers of the first kind $C(n,k)$ denote the number of permutations of $\{1,2,\ldots,n\}$ with exactly k cycles. Derive a recurrence relation for $C(n,k)$. Starting from this recurrence, derive a recurrence relation for the Stirling numbers of the first kind $S_1(n,k) := (-1)^{n-k}C(n,k)$.

16. Let x be an indeterminate and $n \in \mathbb{N}$. Show that

    (a) $\binom{x}{n} = (-1)^n\binom{n-x-1}{n}$

    (b) $x^n = \sum_{k=0}^{n} S_2(n,k)x^{\underline{k}}$

    (c) $x^{\underline{n}} = \sum_{k=0}^{n} S_1(n,k)x^k$