SOFTWARE EXHIBITION

mregular.lib, a SINGULAR 2.0 library for computing the Castelnuovo-Mumford regularity

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In this exhibition, we will present the updated version of the SINGULAR 2.0 library, `mregular.lib` ([6]), jointly written by the two authors and **Gert-Martin Greuel** (University of Kaiserslautern, Germany).

If $R := K[x_0, \ldots, x_n]$ denotes the polynomial ring in $n+1$ variables over an arbitrary field K, the Castelnuovo-Mumford regularity of a homogeneous ideal $I \subset R$, $\operatorname{reg} I$, can be easily obtained from a minimal graded free resolution of I: $\operatorname{reg} I$ is, by definition, the smallest integer m such that $e_{ij} \leq m + i$ for all i, j, where the integers e_{ij} are the degrees of the minimal generators of the i-th syzygy module. The regularity is a numerical invariant of the ideal I and is, as said in [7], "an important measure of how hard it will be to compute a free resolution". In fact, knowing it beforehand avoids unnecessary computation in large degrees while obtaining the minimal graded free resolution of I through Buchberger's syzygy algorithm. The algorithms implemented in this library compute the Castelnuovo-Mumford regularity of an arbitrary homogeneous ideal when the field K is infinite. Variables are assumed to be in Noether position. These algorithms were obtained as consequences of our joint work, [3], [4], and more recently [5]. In these papers, the Castelnuovo-Mumford regularity is computed avoiding the construction of a minimal graded free resolution. Our motivation is [1], and our principal sources of inspiration are [2] and [8].

The version of [6] currently available at `http://www.singular.uni-kl.de` with the software SINGULAR 2.0.3 ([9]) contains the following three procedures:

```
reg_CM(id);          regularity of arith. C-M subscheme V(id_sat) of Pn
reg_curve(id,[,e]); regularity of projective curve V(id_sat) in Pn
reg_moncurve(li);    regularity of projective monomial curve defined by li
```

The algorithms implemented there used the results in [3] and [4], and dealed with the case of a saturated ideal defining either an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n_K$ of arbitrary dimension, or an arbitrary (not necessarily reduced) curve in $\mathbb{P}^n_K$. Since we work with variables in Noether position, the defining ideal of monomial curves are a natural class of ideals where our methods directly apply. That's why our third procedure was dedicated to this special class of ideals.

In the more recent paper [5], the problem of the computation of the Castelnuovo-Mumford regularity is solved for arbitrary homogeneous ideals I in R (saturated or not, and with $\dim R/I$ arbitrary). We will present here the actualized version of our library that includes the following procedures:

```
reg(id);             regularity of homogeneous ideal id
depth(id);           depth of S/id for id homogeneous ideal of basering S
satiety(id);         saturation index of homogeneous ideal id
reg_monvar(li);      regularity of projective monomial variety defined
                     by list of integral vectors li
```

Note that we also compute the depth of R/I, another invariant which is usually obtained from the minimal graded free resolution of I.

In this demo, we will present all these procedures and give a series of non trivial examples. In particular, we will compute in a few seconds, the Castelnuovo-Mumford regularity of a monomial surface in $\mathbb{P}^{11}_{\mathbb{Q}}$ whose minimal graded free resolution could not be obtained using SINGULAR (even Schreyer's resolution, which is easier to compute but only provides an upper bound for the regularity since it is not necessarily minimal, could not be obtained).

References

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