Congruences for k-Elongated Partition Diamonds

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Acknowledgements

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Thanks to Robson da Silva (Universidade Federal de São Paulo, Brazil) and Mike Hirschhorn (University of New South Wales) for an extremely fruitful collaboration.

The results that I will share with you today appear in a paper, co–authored with Robson and Mike, which was recently published in *Discrete Mathematics*.

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- Describe work that Robson, Mike, and I have since completed to prove infinitely many additional congruences for these objects using truly elementary methods
- Close with some thoughts on possible future work

Introductory Thoughts

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In that paper, Andrews and Paule introduced the idea of k-elongated partition diamonds.

Recently, Andrews and Paule revisited the topic of k–elongated partition diamonds, and they published their results in a paper in the *Journal of Number Theory* in 2022.

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Using partition analysis and the Omega operator, Andrews and Paule proved the generating function for the partition numbers $d_k(n)$ produced by summing the links of k–elongated plane partition diamonds of length n.

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$$\sum_{n=0}^{\infty} d_k(n)q^n = \frac{f_2^k}{f_1^{3k+1}}$$

where $f_r = (q^r; q^r)_\infty$ is the usual q–Pochhammer symbol.

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where $f_r = (q^r; q^r)_\infty$ is the usual q–Pochhammer symbol.

They then proceeded to prove several (individual) congruence properties satisfied by d_1, d_2 and d_3 using modular forms and Nicolas Smoot's Mathematica package as their primary proof tools.

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Theorem (Smoot): For all $\alpha \geq 0$ and all $n \geq 0$ such that $8n \equiv 1 \pmod{3^\alpha}$,

$$d_2(n) \equiv 0 \pmod{3^{2\lfloor \alpha/2 \rfloor + 1}}.$$

Introductory Thoughts

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The proof techniques employed below are all elementary, relying on generating function manipulations and classical q-series results.

We require a number of well–known lemmas in order to complete our proofs, some of which I will mention here:

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Lemma:

$$\frac{f_1^5}{f_2^2} = \sum_{m=-\infty}^{\infty} (6m+1) q^{m(3m+1)/2}.$$

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Theorem: (Andrews and Paule, Corollary 5) For all $n \geq 0$, $d_2(3n+2) \equiv 0 \pmod 3$.

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As a first example, we note the following:

Theorem: (Andrews and Paule, Corollary 5) For all $n \geq 0$, $d_2(3n+2) \equiv 0 \pmod{3}$.

As we mentioned earlier, Andrews and Paule used significant tools based on the work of Smoot, which are derived from modular forms, in order to prove their congruences for d_2 and d_3.

Elementary Proofs of Several Congruences from Andrews and Paule

Their proof of this theorem is that

$$g_1 \cdot \sum_{m=0}^{\infty} d_2(3m+2)q^m = 3(8+t)(1728+288t+11t^2),$$

where

$$g_1 = \frac{1}{q^3}\frac{(q;q)_\infty^{19}(q^2;q^2)_\infty(q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^{21}}$$

and

$$t = \frac{1}{q}\frac{(q;q)_\infty^5(q^3;q^3)_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty^5}.$$

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Our proof of this result is very different.

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Proof:

$$\sum_{n=0}^{\infty} d_2(n)q^n = \frac{f_2^2}{f_1^7} = \frac{f_2^2}{f_1}\frac{1}{f_1^6} \equiv \frac{1}{f_3^2}\left(\sum_{m\geq 0} q^{m(m+1)/2}\right) \pmod{3}.$$

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or

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This congruence never holds because 2 is a quadratic nonresidue modulo 3. □

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Theorem: (Andrews and Paule, Corollary 10) For all $n \geq 0$, $d_3(2n+1) \equiv 0 \pmod{2}$.

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Proof: Note that

$$\begin{aligned} \sum_{n=0}^{\infty} d_3(n)q^n &= \frac{f_2^3}{f_1^{10}} \\ &\equiv \frac{f_2^3}{f_2^5} \pmod{2} \\ &\equiv \frac{1}{f_2^2} \pmod{2}. \end{aligned}$$

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Since $\frac{1}{f_2^2}$ is an even function of q, the result follows. □

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Proof: Thanks to the proof of the previous result, we know

$$\begin{aligned} \sum_{n=0}^{\infty} d_3(n) q^n &\equiv \frac{1}{f_2^2} \pmod{2} \\ &\equiv \frac{1}{f_4} \pmod{2} \end{aligned}$$

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$$\equiv \frac{1}{f_4} \pmod{2}$$

Since $\frac{1}{f_4}$ is a function of q^4, the result follows. □

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Theorem: (Andrews and Paule, Corollary 14) For all $n \geq 0$, $d_3(5n+1) \equiv 0 \pmod 5$.

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Proof: We have

$$\begin{aligned}\sum_{n=0}^{\infty} d_3(n)q^n &= \frac{f_2^3}{f_1^{10}} \\ &\equiv \frac{f_2^3}{f_5^2} \pmod{5} \\ &= \frac{1}{f_5^2}\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)}\right).\end{aligned}$$

Elementary Proofs of Several Congruences from Andrews and Paule

We now need to ask whether $5n + 1$ can be represented as $m(m + 1)$, and this is equivalent to asking whether $4(5n + 1) + 1$ or $20n + 5$ can be represented as $(2m + 1)^2$.

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If this is the case, then we know

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which implies that $2m+1 \equiv 0 \pmod 5$.

Thanks to the presence of the coefficient of $2m+1$ in front of the term $q^{m(m+1)}$ in the series above, and the fact that this $2m+1$ must be divisible by 5, we know that our result follows. □

Elementary Proofs of Several Congruences from Andrews and Paule

Theorem: (Andrews and Paule, Corollary 15) For all $n \geq 0$,

$$\begin{aligned} d_3(5n+3) &\equiv 0 \pmod{5}, \\ d_3(5n+4) &\equiv 0 \pmod{5}. \end{aligned}$$

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Theorem: (Andrews and Paule, Corollary 15) For all $n \geq 0$,

$$\begin{aligned} d_3(5n+3) &\equiv 0 \pmod{5}, \\ d_3(5n+4) &\equiv 0 \pmod{5}. \end{aligned}$$

Proof: In the proof of the previous result, we noted that

$$\begin{aligned} &\sum_{n=0}^{\infty} d_3(n)q^n \\ \equiv\ & \frac{1}{f_5^2}\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)}\right) \pmod{5}. \end{aligned}$$

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We now need to ask whether $5n + 3$ can be represented as $m(m + 1)$, and this is equivalent to asking whether $4(5n + 3) + 1$ or $20n + 13$ can be represented as $(2m + 1)^2$.

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This would mean that $(2m+1)^2 \equiv 3 \pmod{5}$.

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However, since 3 is a quadratic nonresidue modulo 5, we know that this cannot be the case.

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However, since 3 is a quadratic nonresidue modulo 5, we know that this cannot be the case.

Similarly, note that

$$4(5n + 4) + 1 = 20n + 17 \equiv 2 \pmod 5$$

and 2 is the other quadratic nonresidue modulo 5. □

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We begin with an unexpected congruence modulo 11 satisfied by the d_2 function.

Theorem: For all $n \geq 0$, $d_2(11n+7) \equiv 0 \pmod{11}$.

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Next, we prove a theorem connecting $d_\ell(n)$ and $p(n)$, where $p(n)$ denotes the number of unrestricted partitions of n.

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Theorem: For all $n \geq 0$,

$$\begin{aligned} d_5(5n+4) &\equiv 0 \pmod{5}, \\ d_7(7n+5) &\equiv 0 \pmod{7}, \\ d_{11}(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

New "Individual" Congruences

Proof: For prime ℓ, the generating function for $d_\ell(n)$ satisfies

$$\sum_{n=0}^{\infty} d_\ell(n)q^n = \frac{f_2^\ell}{f_1^{3\ell+1}} \equiv \frac{f_{2\ell}}{f_\ell^3}\frac{1}{f_1} \pmod{\ell}$$
$$= \frac{f_{2\ell}}{f_\ell^3}\sum_{n=0}^{\infty} p(n)q^n.$$

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$$= \frac{f_{2\ell}}{f_\ell^3}\sum_{n=0}^{\infty} p(n)q^n.$$

Since $\frac{f_{2\ell}}{f_\ell^3}$ is a function of q^ℓ and $p(\ell n + r) \equiv 0 \pmod{\ell}$ for $(\ell, r) = (5,4), (7,5)$, and $(11,6)$, the result follows. □

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Theorem: For all $n \geq 0$,

$$\begin{aligned} d_7(4n+2) &\equiv 0 \pmod{4}, \\ d_7(8n+5) &\equiv 0 \pmod{4}, \\ d_7(16n+9) &\equiv 0 \pmod{4}, \\ d_7(4n+3) &\equiv 0 \pmod{8}, \\ d_7(8n+4) &\equiv 0 \pmod{8}. \end{aligned}$$

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Theorem: For all $n \geq 0$,

$$\begin{aligned} d_8(3n+2) &\equiv 0 \pmod{9}, \\ d_8(9n+3) &\equiv 0 \pmod{9}. \end{aligned}$$

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Said differently (and, potentially, more clearly), we want to demonstrate families of congruences for "fixed" moduli where the subscripts k range over an infinite set (and the arithmetic progressions in question are "fixed" or follow a nice pattern).

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We begin with a somewhat surprising result, primarily because the moduli in question range across **all** primes $p \geq 5$.

New Infinite Families of Congruences

Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$ and $N \geq 1$,

$$d_{p^N-2}(pn+r) \equiv 0 \pmod{p^N}.$$

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Proof: The generating function for $d_{p^N-2}(n)$ satisfies

$$\sum_{n=0}^{\infty} d_{p^N-2}(n)q^n = \frac{f_2^{p^N-2}}{f_1^{3p^N-5}} \equiv \frac{f_1^5}{f_2^2}\frac{f_{2p}^{p^{N-1}}}{f_p^{3p^{N-1}}} \pmod{p^N}.$$

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Thanks to the lemma above from Ramanujan,

$$\sum_{n=0}^{\infty} d_{p^N-2}(n)q^n \equiv \frac{f_{2p}^{p^{N-1}}}{f_p^{3p^{N-1}}} \sum_{m=-\infty}^{\infty} (6m+1)q^{m(3m+1)/2} \pmod{p^N}.$$

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Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be a quadratic nonresidue modulo p. Then, for all $n \geq 0$ and $N \geq 1$,

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We next provide an overarching result which allows us to naturally generalize all of the results we've shared above.

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then for all $n \geq 0$,

$$d_{p^N j+k}(pn + r) \equiv 0 \pmod{p^N}.$$

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In what follows, the notation

$$An + B_1, B_2, \ldots, B_t$$

means we are considering the set of arithmetic progressions

$$An + B_1, An + B_2, \ldots, An + B_t.$$

New Infinite Families of Congruences

Corollary: For all $j \geq 0$ and $n \geq 0$,

$$
\begin{aligned}
d_{2j+1}(2n+1) &\equiv 0 \pmod{2},\\
d_{3j+2}(3n+2) &\equiv 0 \pmod{3},\\
d_{5j+3}(5n+1,3,4) &\equiv 0 \pmod{5},\\
d_{5j+4}(5n+2,3) &\equiv 0 \pmod{5},\\
d_{5j+5}(5n+4) &\equiv 0 \pmod{5},\\
d_{7j+5}(7n+2,3,4,6) &\equiv 0 \pmod{7},\\
d_{7j+6}(7n+3,5,6) &\equiv 0 \pmod{7},\\
d_{7j+7}(7n+5) &\equiv 0 \pmod{7},\\
d_{11j+2}(11n+7) &\equiv 0 \pmod{11},\\
d_{11j+9}(11n+3,5,6,8,9,10) &\equiv 0 \pmod{11},\\
d_{11j+10}(11n+2,6,7,8,10) &\equiv 0 \pmod{11},\\
d_{11j+11}(11n+6) &\equiv 0 \pmod{11},\\
d_{13j+11}(13n+3,4,6,7,8,10,11) &\equiv 0 \pmod{13},\\
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Our goal in writing these here is to provide a representative set of the kinds of congruences that arise within this family of partition functions.

Closing Questions/Thoughts

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- $d_7(n)$ modulo powers of 2
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- $d_{15}(n)$ modulo powers of 2

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Indeed, for $k = 2^j - 1$ for some j, it is clear that the generating function for $d_k(n)$ will have a structure that allows for a number of congruences to hold for small powers of 2.

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Let me also return to one of the first congruences I mentioned in this talk:

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Let me also return to one of the first congruences I mentioned in this talk:

Theorem: (Andrews and Paule, Corollary 5) For all $n \geq 0$, $d_2(3n+2) \equiv 0 \pmod 3$.

Closing Questions/Thoughts

Andrews and Paule's proof of this theorem is that

$$g_1 \cdot \sum_{m=0}^{\infty} d_2(3m+2)q^m = 3(8+t)(1728+288t+11t^2),$$

where

$$g_1 = \frac{1}{q^3}\frac{(q;q)_\infty^{19}(q^2;q^2)_\infty(q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^{21}}$$

and

$$t = \frac{1}{q}\frac{(q;q)_\infty^5(q^3;q^3)_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty^5}.$$

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A few minutes ago, I noted that this result is the first case of a much larger family of congruences, namely,

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Theorem: (da Silva, Hirschhorn, and JAS) For all $j \geq 0$ and $n \geq 0$, $d_{3j+2}(3n+2) \equiv 0 \pmod{3}$.

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Would there be a pattern to the proofs, so that one unified proof could be written down to prove the result for all $j \geq 0$ simultaneously?

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Lastly, what can be said about similar proofs (based on modular forms) for all of the other families of congruences I have demonstrated today?

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Lastly, what can be said about similar proofs (based on modular forms) for all of the other families of congruences I have demonstrated today?

And with that I will close. Thanks very much for attending today!

Congruences for k-Elongated Partition Diamonds

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