Plethysm and the algebra of uniform block permutations

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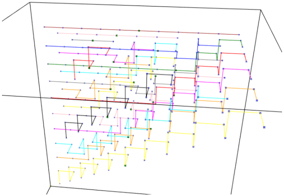
based on joint work with Rosa Orellana (Dartmouth), Franco Saliola (UQAM), Mike Zabrocki (York), Algebraic Combinatorics (2022), to appear
OSZ, Laura Colmenarejo (NCSU) in progress



Algorithmic and Enumerative Combinatorics
Vienna
July 5, 2022

FPSAC 2023 at UC Davis: July 17-21, 2023

UCD

FPSAC

Davis, California **2023**

fpsac23.math.ucdavis.edu

Outline

1. The plethysm problem
2. Diagram algebras
3. Uniform block permutation algebra

Representations

G group, V vector space

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- Representation $\rho\colon G \to \mathrm{End}(V)$ homomorphism

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Remark

Characters are class functions, that is, they are constant on conjugacy classes $\mathrm{char}(hgh^{-1}) = \mathrm{char}(g)$.

Plethysm via representations of GL_n

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$GL_n(\mathbb{C})$ = invertible $n \times n$ matrices

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Definition

Character of composition is plethysm:

$$\mathrm{char}(\tau \circ \rho) = \mathrm{char}(\tau)[\mathrm{char}(\rho)]$$

Frobenius map

R^n space of class functions of GL_n
Λ^n ring of symmetric functions of degree n

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$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$$
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$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}$$

Hall inner product

$$\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

Frobenius map – continued

Definition

The Frobenius characteristic map is $\mathrm{ch}^n\colon R^n \to \Lambda^n$

$$\mathrm{ch}^n(\chi) = \sum_{\mu \vdash n} \frac{1}{z_\mu} \chi_\mu p_\mu$$

where $z_\mu = 1^{a_1} a_1! 2^{a_2} a_2! \cdots$ for $\mu = 1^{a_1} 2^{a_2} \cdots$

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Remark

The irreducible character χ^λ indexed by λ under the Frobenius map is

$$\mathrm{ch}^n(\chi^\lambda) = s_\lambda$$

by the identity

$$s_\lambda = \sum_{\mu} \frac{1}{z_\mu} \chi^\lambda_\mu p_\mu$$

Plethysm for symmetric functions

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Monomial expansion $f = \sum_{i \geqslant 1} x^{a^i}$

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$s_1 = x_1 + x_2 + \cdots \qquad \Rightarrow \qquad g[s_1] = g(x_1, x_2, \ldots) = g$

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$s_1 = x_1 + x_2 + \cdots \quad \Rightarrow \quad g[s_1] = g(x_1, x_2, \dots) = g$

$p_n = x_1^n + x_2^n + \cdots \quad \Rightarrow \quad f[p_n] = f(x_1^n, x_2^n, \dots) = \sum_{i \geqslant 1} x^{a^i n} = p_n[f]$

Plethysm for symmetric functions – example

Example

$$s_2[x_1, x_2] = x_1^2 + x_1 x_2 + x_2^2$$

$$\boxed{1|1} \quad \boxed{1|2} \quad \boxed{2|2}$$

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Plethysm

$$s_2[s_2[x_1, x_2]] = s_2[x_1^2, x_1x_2, x_2^2]$$
$$= x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4$$

$$\boxed{1|1} \quad \boxed{1|2} \quad \boxed{1|3} \quad \boxed{2|2} \quad \boxed{2|3} \quad \boxed{3|3}$$

$$\boxed{11|11} \quad \boxed{11|12} \quad \boxed{11|22} \quad \boxed{12|12} \quad \boxed{12|22} \quad \boxed{22|22}$$

$$= s_4[x_1, x_2] + s_{2,2}[x_1, x_2]$$

Plethysm problem

Problem

Find a combinatorial interpretation for the coefficients $a^{\nu}_{\lambda\mu} \in \mathbb{N}$ *in the expansion*

$$s_\lambda[s_\mu] = \sum_{\nu} a^{\nu}_{\lambda\mu} s_\nu$$

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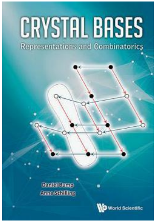
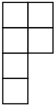
Problem

Find a crystal on tableaux of tableaux which explains $a^{\nu}_{\lambda\mu}$.



Plethysm problem – special cases

Partition λ is even if all columns have even length 

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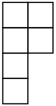
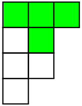
Partition λ is even if all columns have even length 

Partition λ is threshold if $\lambda'_i = \lambda_i + 1$ for all $1 \leqslant i \leqslant d(\lambda)$ 

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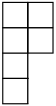
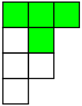
Partition λ is even if all columns have even length 

Partition λ is threshold if $\lambda'_i = \lambda_i + 1$ for all $1 \leqslant i \leqslant d(\lambda)$ 

Theorem

We have

$$s_h[s_2] = \sum_{\substack{\lambda \vdash 2h \\ \lambda \text{ even}}} s_{\lambda'} \qquad s_h[s_{1^2}] = \sum_{\substack{\lambda \vdash 2h \\ \lambda \text{ even}}} s_{\lambda}$$

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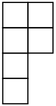
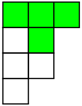
Partition λ is even if all columns have even length 

Partition λ is threshold if $\lambda'_i = \lambda_i + 1$ for all $1 \leqslant i \leqslant d(\lambda)$ 

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Appeared in Littlewood 1950, Macdonald 1998 (pg 138)

Littlewood and Macdonald

Easy proof – s-perp trick

Action of $s_\lambda^\perp$ on $f \in \Lambda$

$$s_\lambda^\perp f = \sum_\mu \langle f, s_\lambda s_\mu \rangle s_\mu$$

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Proposition (The s-perp trick)

Let f and g be two symmetric functions of homogeneous degree d. If

$$s_r^\perp f = s_r^\perp g \qquad \text{for all } 1 \leqslant r \leqslant d,$$

then $f = g$. Same statement is true if $s_r^\perp$ is replaced by $s_{1^r}^\perp$.

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The following hold:

$$s_r^\perp s_{1^h}[s_{1^w}] = s_{1^{h-r}}[s_{1^w}]s_{1^r}[s_{1^{w-1}}] \qquad s_{1^r}^\perp s_h[s_w] = s_{h-r}[s_w]s_{1^r}[s_{w-1}]$$

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Remark

Benefit: Fast computational algorithm to compute plethysm of Schur functions!

Relationship between restriction problem and plethysm

Restriction: λ partition with at most n parts

$$\mathrm{Res}_{S_n}^{GL_n} V_{GL_n}^{\lambda} = \bigoplus \left(V_{S_n}^{\mu}\right)^{r_{\lambda\mu}}$$

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$r_{\lambda\mu}$ = coefficient of s_μ in the plethysm $s_{(n-|\lambda|,\lambda)}[s_{(1)} + s_{(2)} + \cdots]$

Outline

1. The plethysm problem
2. Diagram algebras
3. Uniform block permutation algebra

Diagram algebras

- Restrict diagonal action of GL_n on $V^{\otimes k}$ to $S_n \subseteq GL_n$: for $\sigma \in S_n$

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sigma v_{i_1} \otimes \cdots \otimes \sigma v_{i_k}$$

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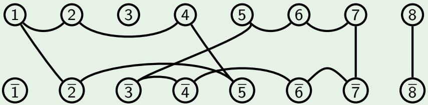
Example

The set partition $\pi = \{\{1, 2, 4, \overline{2}, \overline{5}\}, \{3\}, \{5, 6, 7, \overline{3}, \overline{4}, \overline{6}, \overline{7}\}, \{8, \overline{8}\}, \{\overline{1}\}\}$ is represented by the following diagram:

$\pi =$



Martin and Jones

Centralizer pair

$V_{P_k(n)}^{(n-|\lambda|,\lambda)}$ = simple module indexed by partitions λ such that $\lambda_1 + \lambda_2 + \cdots \leqslant k$

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Theorem (Jones 1994)

$$V^{\otimes k} \cong \bigoplus_{\lambda, \lambda_1+\lambda_2+\cdots \leqslant k} V_{P_k(n)}^{(n-|\lambda|,\lambda)} \otimes V_{S_n}^{(n-|\lambda|,\lambda)}$$

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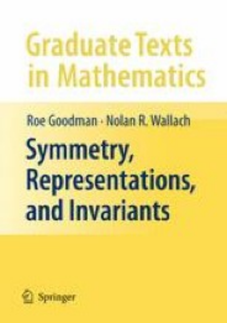
Remark

- S_k and GL_n form a centralizer pair
- $P_k(n)$ and S_n form a centralizer pair

See-Saw pairs



(See book by Goodman, Wallach)

See-Saw pairs

$A \hookrightarrow B$ algebra embedding $\qquad \mathrm{Res}^B_A\, V^\lambda_B = \bigoplus_\mu \left(V^\mu_A\right)^{\oplus c_{\lambda\mu}}$

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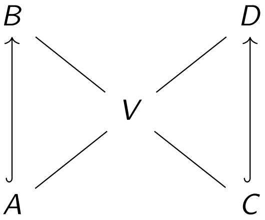

- B and C centralizer pair
- A and D centralizer pair

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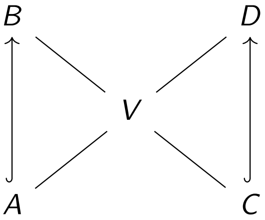

- B and C centralizer pair
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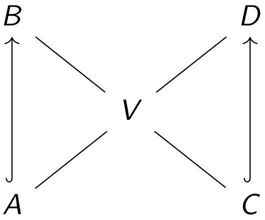
1. Indices for the simple modules for B and C are the same.
2. Indices for the simple modules for A and D are the same.

See-Saw pairs

$A \hookrightarrow B$ algebra embedding $\qquad \mathrm{Res}^B_A\ V^\lambda_B = \bigoplus_\mu \left(V^\mu_A\right)^{\oplus c_{\lambda\mu}}$



- B and C centralizer pair
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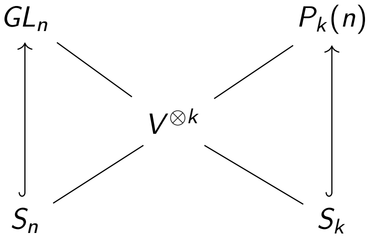
$$\mathrm{Res}^D_C\ V^\mu_D = \bigoplus_\lambda \left(V^\lambda_C\right)^{\oplus c_{\lambda\mu}}$$

Our See-Saw pair



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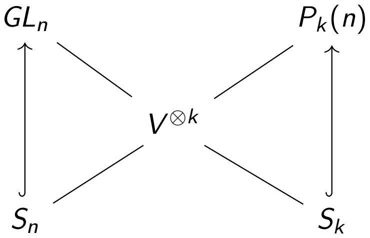
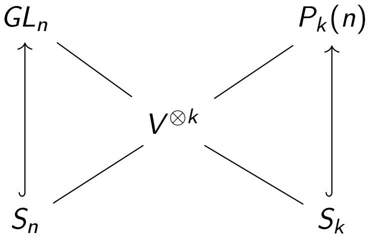


$$\mathrm{Res}_{S_n}^{GL_n} V_{GL_n}^{\lambda} = \bigoplus_{\mu} \left(V_{S_n}^{\mu}\right)^{\oplus r_{\lambda\mu}}$$

$$\mathrm{Res}_{S_k}^{P_k(n)} V_{P_k(n)}^{\mu} = \bigoplus_{\lambda} \left(V_{S_k}^{\lambda}\right)^{\oplus r_{\lambda\mu}}$$

Our See-Saw pair



$$\mathrm{Res}^{GL_n}_{S_n} \; V^{\lambda}_{GL_n} = \bigoplus_{\mu} \left(V^{\mu}_{S_n}\right)^{\oplus r_{\lambda\mu}}$$

$$\mathrm{Res}^{P_k(n)}_{S_k} \; V^{\mu}_{P_k(n)} = \bigoplus_{\lambda} \left(V^{\lambda}_{S_k}\right)^{\oplus r_{\lambda\mu}}$$

Idea: Restrict representations of $P_k(n)$ to S_k

The approach

$\mathcal{U}_k$ uniform block permutation algebra

$$\underbrace{S_k \hookrightarrow}_{\text{special cases of plethysm}} \mathcal{U}_k \underbrace{\hookrightarrow P_k(n)}_{\text{generalized LR coefficients}}$$

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$\mathcal{U}_k$ uniform block permutation algebra

$$\underbrace{S_k \hookrightarrow}_{\text{special cases of plethysm}} \mathcal{U}_k \underbrace{\hookrightarrow P_k(n)}_{\text{generalized LR coefficients}}$$

Goal: Combinatorial model for the representation theory of $\mathcal{U}_k$

Outline

1 The plethysm problem

2 Diagram algebras

3 Uniform block permutation algebra

Uniform block permutations

Tanabe, Kosuda

Party algebra, centralizer algebra for complex reflection groups

Uniform block permutations

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Party algebra, centralizer algebra for complex reflection groups

Definition

The set partition $d = \{d_1, d_2, \ldots, d_\ell\}$ of $[k] \cup [\bar{k}]$ is uniform if $|d_i \cap [k]| = |d_i \cap [\bar{k}]|$ for all $1 \leqslant i \leqslant \ell$. Let

$$\mathcal{U}_k = \left\{ d \vdash [k] \cup [\bar{k}] : d \text{ uniform} \right\}.$$

Uniform block permutations

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$$\mathcal{U}_k = \left\{ d \vdash [k] \cup [\bar{k}] : d \text{ uniform} \right\}.$$

Example

$$d = \{\{2, \bar{4}\}, \{5, \bar{7}\}, \{1, 3, \bar{1}, \bar{2}\}, \{4, 6, \bar{3}, \bar{6}\}, \{7, 8, 9, \bar{5}, \bar{8}, \bar{9}\}\}$$

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Think of d as a size-preserving bijection

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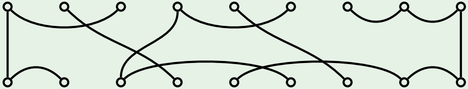
$\Rightarrow$ Elements of $\mathcal{U}_k$ are called uniform block permutations

Uniform block permutations – continued

Example

Diagram for $\{\{1,3,\bar{1},\bar{2}\},\{2,\bar{4}\},\{4,6,\bar{3},\bar{6}\},\{5,\bar{7}\},\{7,8,9,\bar{5},\bar{8},\bar{9}\}\}$



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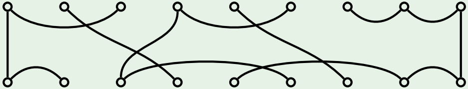
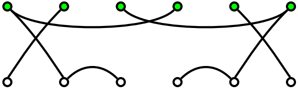
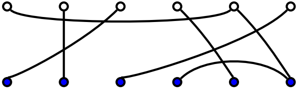
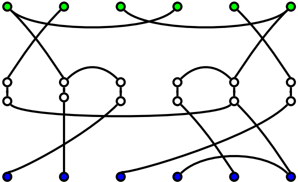
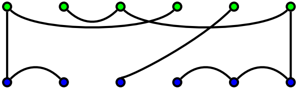


The product of

$d =$  and $d' =$ 

is obtained by stacking the diagrams of d and d':

$dd' =$  $=$ 

Idempotents

For every set partition π of $[k]$ we define:

$$e_\pi = \{A \cup \bar{A} : A \in \pi\} \in \mathcal{U}_k$$

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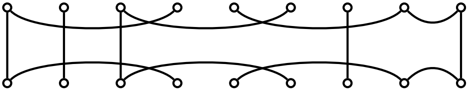
$e_{2|7|14|36|589}$ = 

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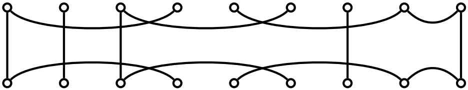
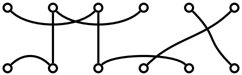


Lemma

The set $E(\mathcal{U}_k) = \{e_\pi : \pi \vdash [k]\}$ *is a complete set of idempotents in* $\mathcal{U}_k$.

Factorizable monoid



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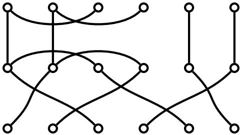

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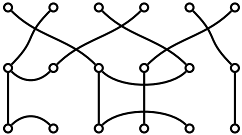

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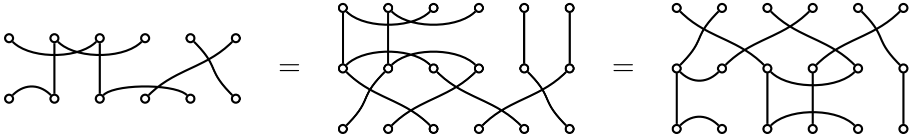


Proposition

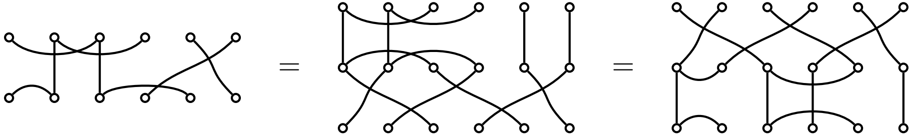
For every $d \in \mathcal{U}_k$ and every $\sigma \in S_k$ satisfying $\sigma(B \cap [k]) = \overline{B} \cap [k]$, we have

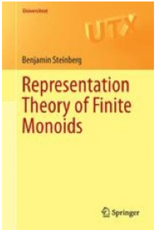
$$d = e_{\mathrm{top}(d)}\, \sigma = \sigma\, e_{\mathrm{bot}(d)}.$$

Consequently, $\mathcal{U}_k$ is a factorizable monoid

$$\mathcal{U}_k = E(\mathcal{U}_k)\, S_k = S_k\, E(\mathcal{U}_k).$$

Factorizable monoid





(See book by Steinberg 2016)

Maximal subgroups

Definition

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Maximal subgroup: $G_e =$ unique largest subgroup of M containing e

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The maximal subgroup of $\mathcal{U}_k$ at the idempotent e_π is

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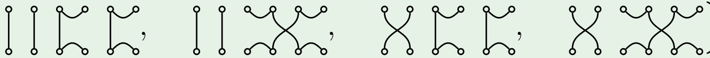
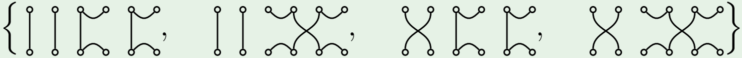
$G_{e_\pi} = \{$  $\}$

Maximal subgroups – continued

Example

For $\pi = \{\{1\},\{2\},\{3,4\},\{5,6\}\}$ with $\text{type}(\pi) = (1^2 2^2)$

$G_{e_\pi} =$



Theorem

For $\pi \vdash [k]$ *with* $\text{type}(\pi) = (1^{a_1} 2^{a_2} \dots k^{a_k})$

$$G_{e_\pi} \simeq S_{a_1} \times S_{a_2} \times \cdots \times S_{a_k}$$

Representation theory of $\mathcal{U}_k$

Indexing set of simple modules

$$I_k = \left\{ \left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}\right) : \lambda^{(i)} \text{ are partitions such that } \sum_{i=1}^{k} i|\lambda^{(i)}| = k \right\}$$

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Example

$I_3 = \{((3), \emptyset, \emptyset), ((2,1), \emptyset, \emptyset), ((1,1,1), \emptyset, \emptyset), ((1), (1), \emptyset), (\emptyset, \emptyset, (1))\}$

Representation theory of $\mathcal{U}_k$ – continued

Definition

A uniform tableau $\mathbf{S} = (S^{(1)}, \ldots, S^{(k)})$ of shape $\vec{\lambda} \in I_k$ satisfies:

1. $S^{(i)}$ is a tableau of shape $\lambda^{(i)}$ filled with subsets of $[k]$ of size i;
2. $S^{(i)}$ is standard;
3. the subsets appearing in $\mathbf{S}$ form a set partition of $[k]$.

We define $\mathcal{T}_{\vec{\lambda}}$ to be the set of uniform tableaux of shape $\vec{\lambda}$.

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$$V_{\mathcal{U}_3}^{((1),(1),\emptyset)} = \operatorname{span}\left\{\left(\boxed{1}, \boxed{23}\right), \left(\boxed{2}, \boxed{13}\right), \left(\boxed{3}, \boxed{12}\right)\right\}$$

Characters of $\mathcal{U}_k$

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M be a finite monoid.

- Subsemigroup of M generated by $m \in M$ contains a unique idempotent m^ω
- $m, n \in M$ are conjugate if there exist $x, x' \in M$ such that $xx'x = x$, $x'xx' = x'$, $x'x = m^\omega$, $xx' = n^\omega$ and $xm^{\omega+1}x' = n^{\omega+1}$

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$d_{\vec{\mu}}$ representative for generalized conjugacy class of cycle type $\vec{\mu}$

Characters of $\mathcal{U}_k$ – continued

Theorem (OSSZ 2022)

$\vec{\lambda}, \vec{\mu} \in I_k$, $a_i = |\lambda^{(i)}|$, $\lambda = (1^{a_1} 2^{a_2} \cdots k^{a_k})$

$$\chi^{\vec{\lambda}}_{\mathcal{U}_k}(d_{\vec{\mu}}) = \sum_{\substack{\vec{\nu} \in I_k \\ |\nu^{(i)}| = a_i}} b^{\vec{\nu}}_{\vec{\mu}} \, \chi^{\vec{\lambda}}_{G_\lambda}(d_{\vec{\nu}})$$

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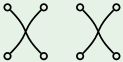
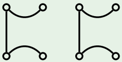
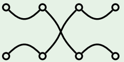
Example

Let $\vec{\lambda} = (\emptyset, (1,1), \emptyset, \emptyset)$, so that $\lambda = (2,2)$:

$\chi^{\vec{\lambda}}_{\mathcal{U}_4}\Bigg($  $\Bigg) = \chi^{\vec{\lambda}}_{G_\lambda}\Bigg($  $\Bigg) + 2\chi^{\vec{\lambda}}_{G_\lambda}\Bigg($  $\Bigg) = -1$

Coefficients in characters

$z_\lambda = 1^{a_1} a_1! 2^{a_2} a_2! \cdots k^{a_k} a_k!$ for $\lambda = (1^{a_1} 2^{a_2} \cdots k^{a_k})$

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Theorem (OSSZ 2022)

$\vec{\mu}, \vec{\nu} \in I_k$

$$b_{\vec{\mu}}^{\vec{\nu}} = \frac{1}{\mathbf{z}_{\vec{\nu}}} \sum_{\vec{\tau}(\bullet,\bullet)} \frac{\mathbf{z}_{\vec{\mu}}}{\prod_{i,j} \mathbf{z}_{\vec{\tau}(i,j)}}$$

where sum is over all $\vec{\tau}(\bullet, \bullet)$ *with* $\vec{\tau}(i,j) \in I_j$ *and* $\vec{\mu} = \biguplus_{i,j} \nu_i^{(j)} \vec{\tau}(i,j)$.

Connections to symmetric functions

Symmetric functions on multiple variables: $\mathbf{X} = X_1, X_2, \ldots$

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Scalar product

$$\langle \mathbf{p}_{\vec{\lambda}}[\mathbf{X}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \rangle = \begin{cases} \mathbf{z}_{\vec{\mu}} & \text{if } \vec{\lambda} = \vec{\mu} \\ 0 & \text{else} \end{cases}$$

Connections to symmetric functions – continued

Frobenius characteristic of trivial representation of $\mathcal{U}_k$

$$E_r := \sum_{\vec{\mu}\in I_r} \frac{\mathbf{p}_{\vec{\mu}}[\mathbf{X}]}{\mathbf{z}_{\vec{\mu}}} = \sum_{(1^{a_1}2^{a_2}\cdots r^{a_r})\vdash r} s_{a_1}[X_1]s_{a_2}[X_2]\cdots s_{a_r}[X_r]$$

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Proposition (OSSZ 2022)

$$b_{\vec{\mu}}^{\vec{\nu}} = \frac{1}{\mathbf{z}_{\vec{\nu}}} \left\langle \mathbf{p}_{\vec{\nu}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle$$

Characters, symmetric functions, and plethysm

Theorem (OSSZ 2022)

$$\chi^{\vec{\lambda}}_{\mathcal{U}_k}(d_{\vec{\mu}}) = \left\langle \mathbf{s}_{\vec{\lambda}}[\mathbf{E}], \mathbf{p}_{\vec{\mu}}[\mathbf{X}] \right\rangle$$

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Corollary

Multiplicity of $V^{\mu}_{S_k}$ *in* $\mathrm{Res}^{\mathcal{U}_k}_{S_k} V^{\vec{\lambda}}_{\mathcal{U}_k}$ *is* $\langle s_{\lambda^{(1)}}[s_1] s_{\lambda^{(2)}}[s_2] \cdots s_{\lambda^{(k)}}[s_k],\ s_\mu \rangle$

Thank you !

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Remark (Take away)

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