Chordal graphs with bounded tree-width

joint work with Jordi Castellví, Michael Drmota and Marc Noy

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TU Wien - 07/07/2022

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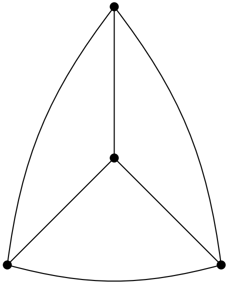
- start with K_{k+1},

Example $(k = 3)$:



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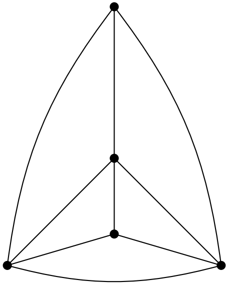

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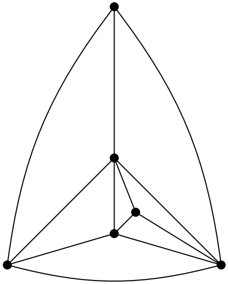

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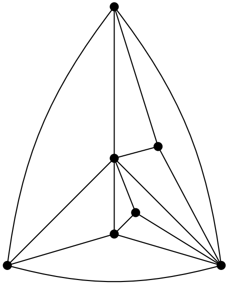

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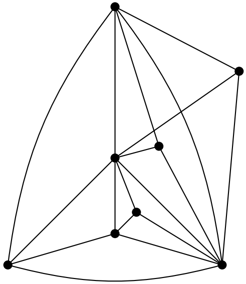

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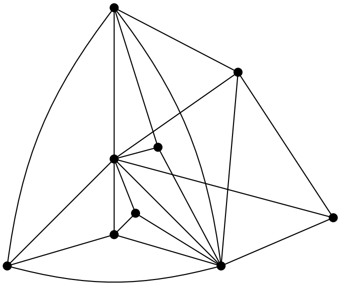

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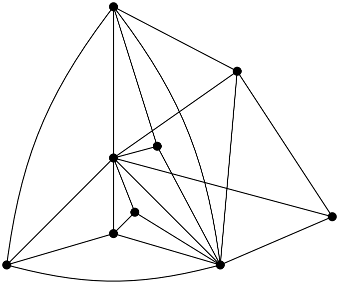

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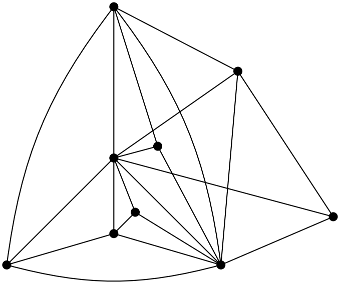

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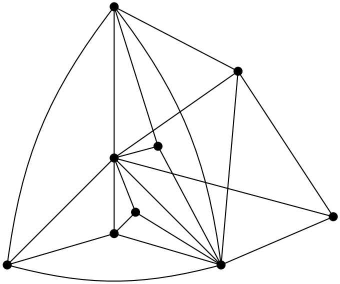

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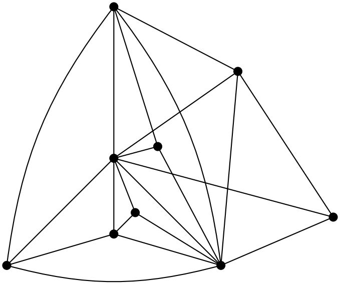

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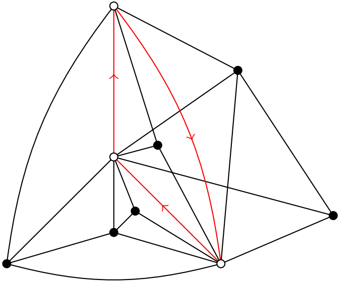
[Beineke & Pippert (1969)]: # of k-trees with n vertices $= \binom{n}{k}(kn - k^2 + 1)^{n-k-2}$

Rooted k-trees

A (labelled) k-tree is rooted when one k-clique is distinguished:

- fix a k-clique and fix an ordering of its vertices then remove their labels



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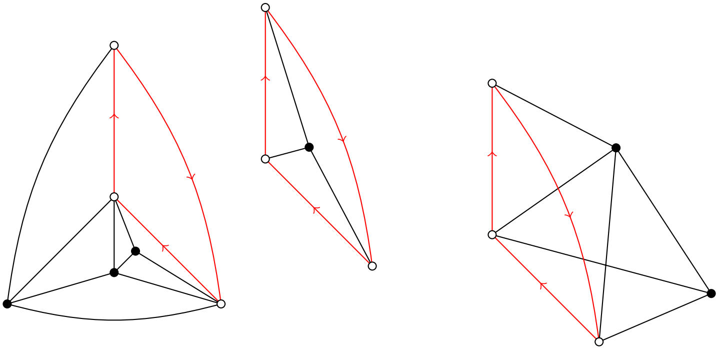

Recursive (implicit) definition of the exponential generating function of rooted k-trees:

$$T_k(x) = \exp\left(xT_k(x)^k\right)$$

Analytic combinatorics: first principle

Exponential growth of the coefficients is determined by the radius of convergence.

$$[x^n]T_k(x) \propto \rho^{-n}, \qquad \text{if } \rho > 0.$$

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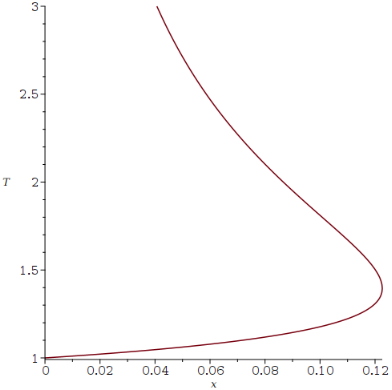

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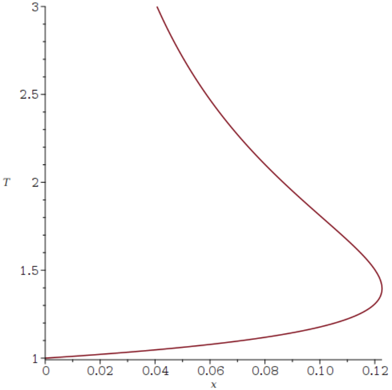

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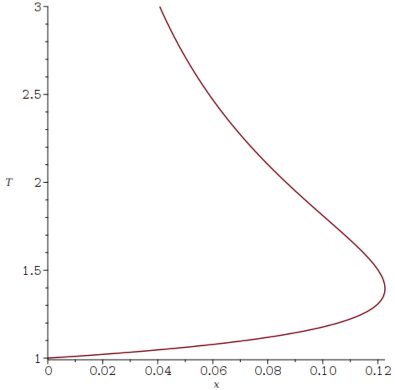

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$$x = \frac{1}{kT_k(x)^k} \implies T_k(x) = \exp(1/k)$$

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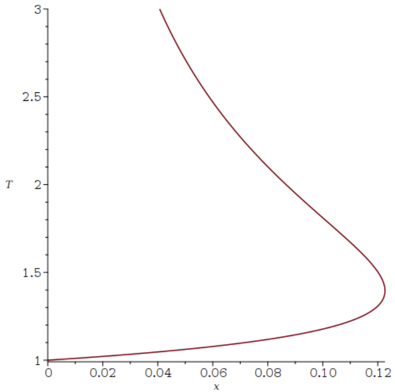

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Radius of convergence of $T_k(z)$ is at $x = (ke)^{-1}$ $\quad\to\quad$ $((3e)^{-1} \approx 0.1226)$.

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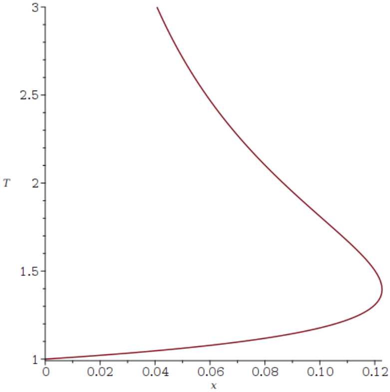
Sub-exponential growth of the coefficients is determined by the behaviour locally around the singularity (the singular expansion).



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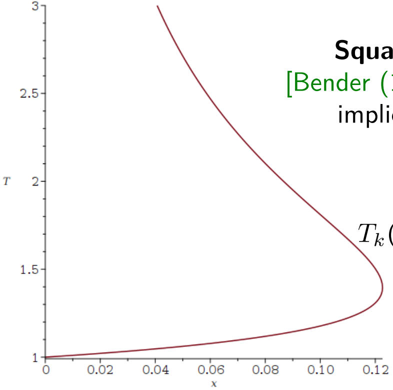

Square-root singular expansion lemma

[Bender (1974)]: Provided "nice" properties of the implicit function $T_k(x) = \exp\left(xT_k(x)^k\right)$, then as $z \sim (ke)^{-1}$

$$T_k(x) = T_0(k) - T_1(k)\left(1 - \frac{x}{(ke)^{-1}}\right)^{1/2} + O\left(1 - \frac{x}{(ke)^{-1}}\right)$$

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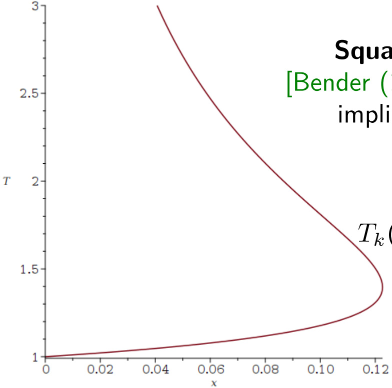

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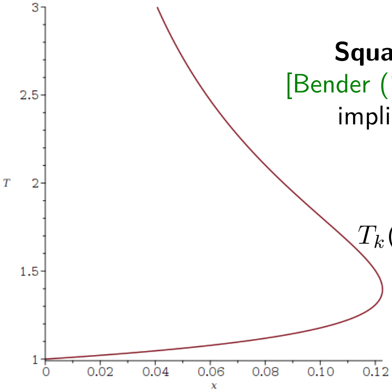

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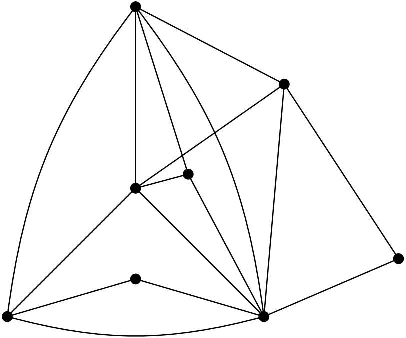
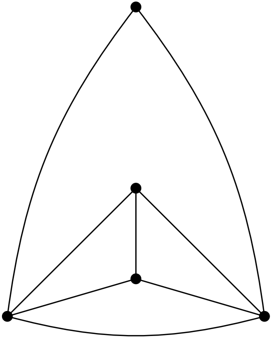
- asymptotic for unrooted k-trees → subexp. term in $n^{-5/2}$

Graphs with bounded tree-width

Graphs with tree-width at most k are exactly the subgraphs of k-trees

- thus called partial k-trees





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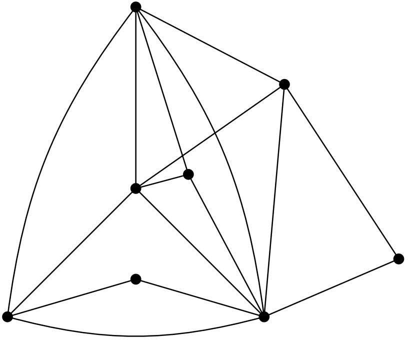

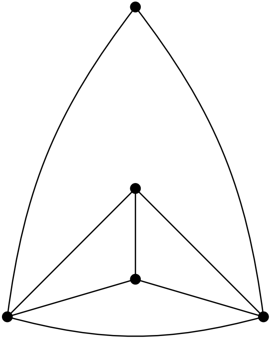

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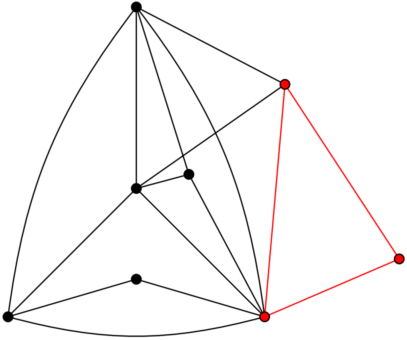

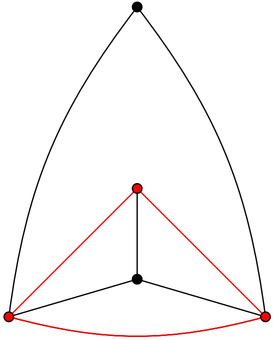

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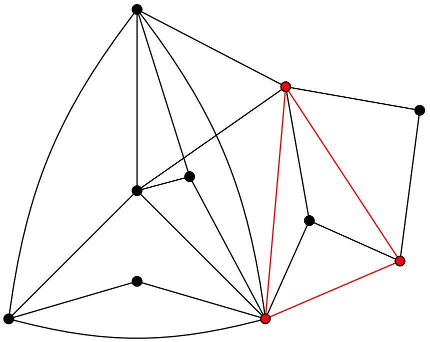

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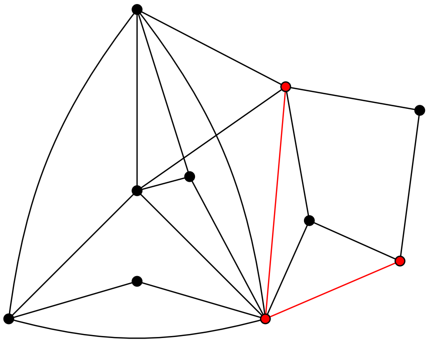

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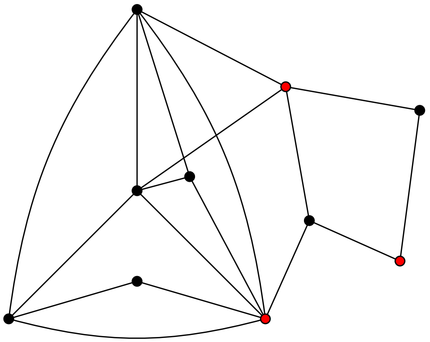

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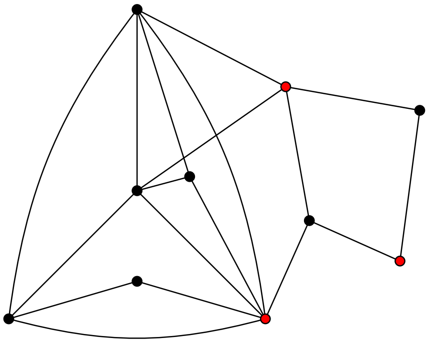


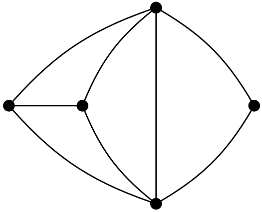
Enumeration: Let g_n be the # of graphs of tree-width at most k with n vertices

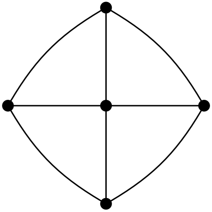
[Baste, Noy & Sau (2018)]: for fixed k and as $n \to \infty$

$$\left(\frac{k}{\log k}\right)^n 2^{nk} n^n \le g_n \le (ek)^n 2^{nk} n!$$

Chordal graphs

Definition: a graph is chordal if it admits no induced cycle of length ≥ 4.





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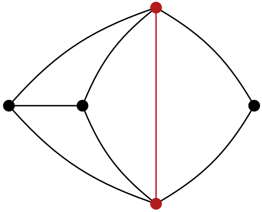

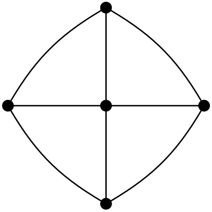

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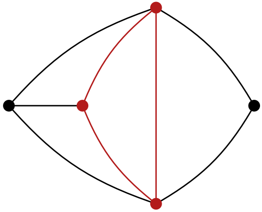

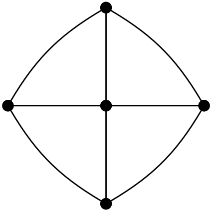

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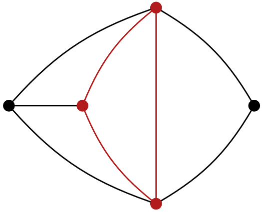

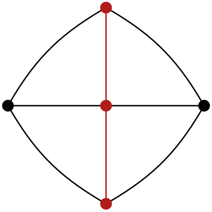

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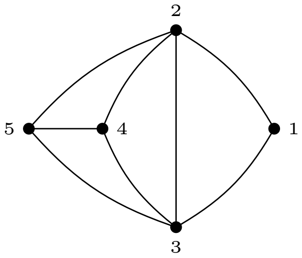

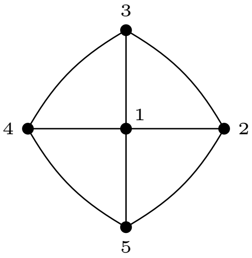

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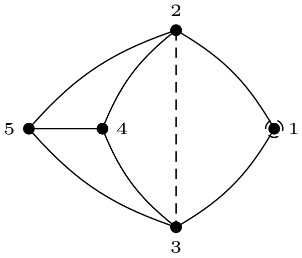

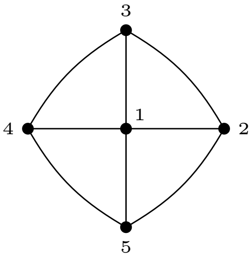

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Chordal graphs

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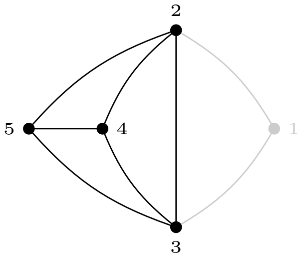

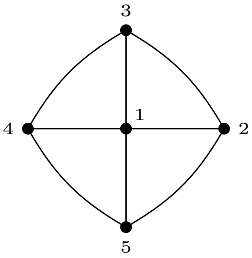

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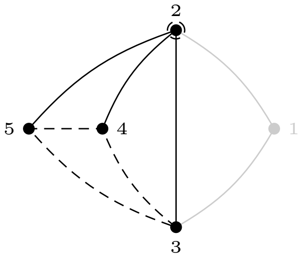

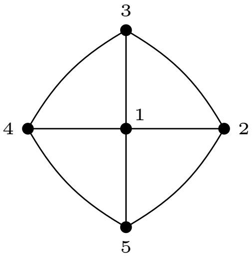

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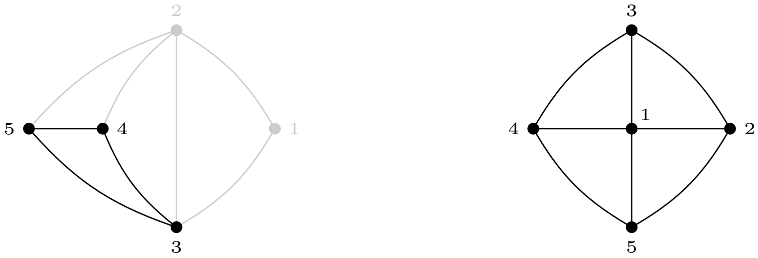

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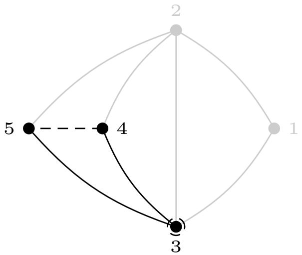

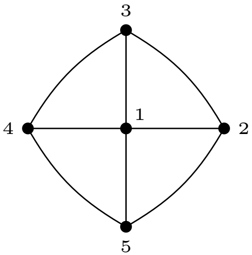

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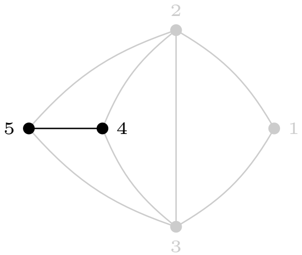

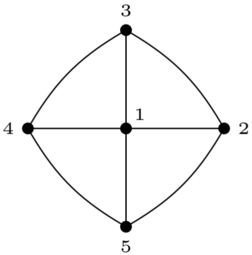

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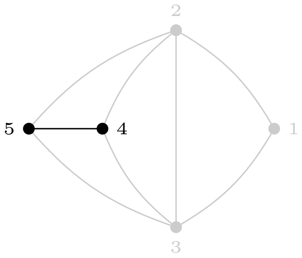

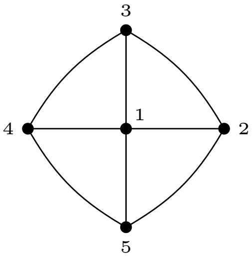

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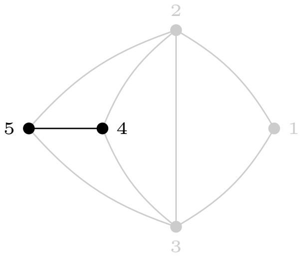

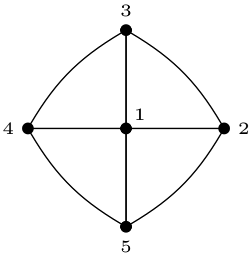

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- k-trees are chordal → k-connected chordal graphs of tree-width $\leq k$,
- when taking the clique-sum of two chordal graphs → no edge removal!

Chordal graphs with bounded tree-width

Fix $n, k \geq 1$ and $0 \leq q \leq k$.

Let $\mathcal{G}_{k,q,n}$ be the family of q-connected chordal graphs with n labelled vertices and tree-width at most k.

[Castellví, Drmota, Noy & R. (2022+)]: $\exists\ c_{k,q} > 0$ and $\gamma_{k,q} \in (0,1)$ s.t.

$$|\mathcal{G}_{k,q,n}| \sim c_{k,q} \cdot n^{-5/2} \cdot \gamma_{k,q}^{n} \cdot n! \qquad \text{as } n \to \infty.$$

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For $i \in [k]$, let X_i = # of i-cliques in a uniform random graph in $\mathcal{G}_{t,k,n}$.

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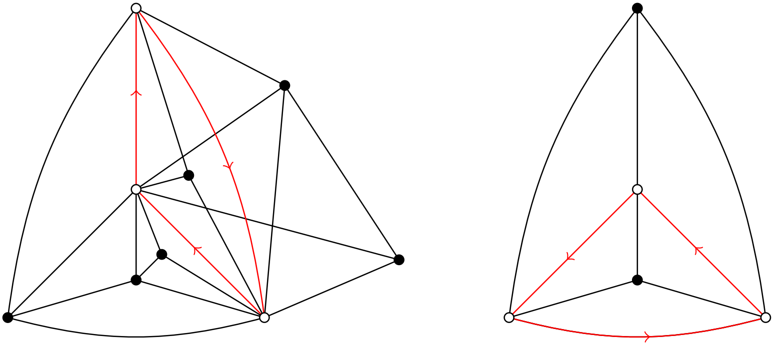
$$\frac{|X_i - \mathbb{E}X_i|}{\sqrt{\mathbb{V}X_i}} \xrightarrow{d} N(0,1), \qquad \text{with} \quad \mathbb{E}X_i \sim \alpha n \quad \text{and} \quad \mathbb{V}X_i \sim \beta n.$$

The $(k-1)$-connected graphs

$$G_{k-1}(x_1, x_{k-1}) = \sum_{A \in \mathcal{G}_{k,k-1}} \frac{x_1^{n_1(A)}}{n_1(A)!} x_{k-1}^{n_{k-1}(A)}, \quad n_j(A) = \# \text{ of } j\text{-cliques of } A, \forall j \in [k]$$



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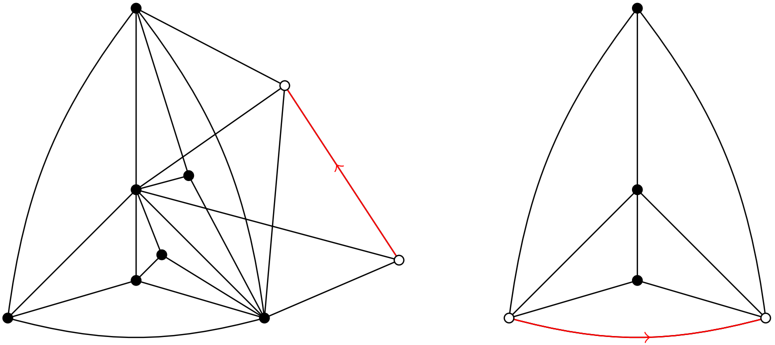

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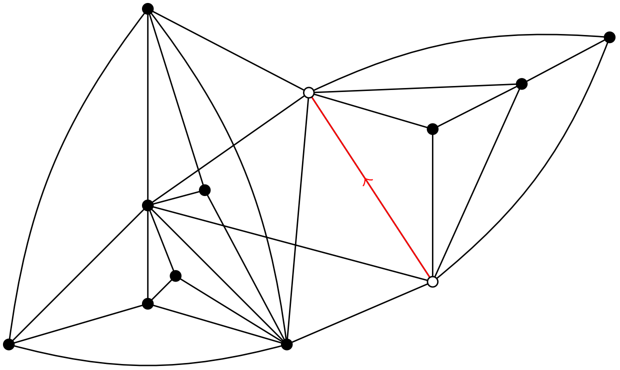

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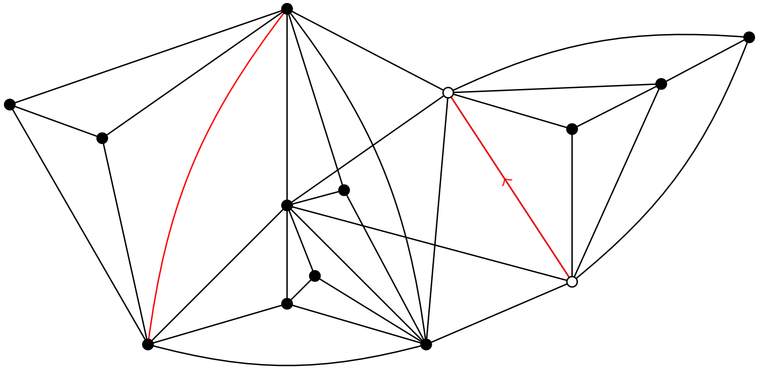

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Down the stairs

Multivariate GF of q-connected graphs: for any $q \in [k]$

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Implict equation for the GF of q-connected graphs rooted at a q-clique

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[Wormald (1985)]: *algorithm to compute the GF of chordal graphs.*

Chordal graphs with small tree-width

k	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q=7$
1	0.36788	-	-	-	-	-	-
2	0.14665	0.18394	-	-	-	-	-
3	0.07703	0.08421	0.12263	-	-	-	-
4	0.04444	0.04662	0.05664	0.09197	-	-	-
5	0.02657	0.02732	0.03092	0.04152	0.07358	-	-
6	0.01608	0.01635	0.01773	0.02184	0.03214	0.06131	-
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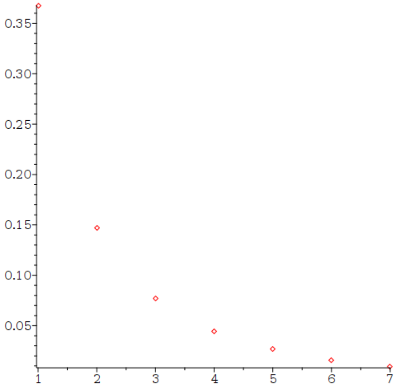

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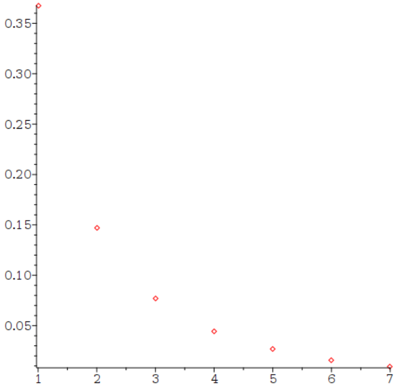

[Bender, Richmond & Wormald (1985)]:
almost all chordal graphs are split.

- # of (labelled) chordal graphs with n vertices is

$$\sim \binom{n}{n/2} 2^{n^2/4}$$

Conclusion

Our results for chordal graphs with tree-width $\leq k$:

- enumerative formula of the form $c_k \cdot n^{-5/2} \rho_k^{-n} n!$
- CLT for the number of i-cliques, for $i \in [k]$
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Danke!