Coefficientwise total positivity of some matrices defined by linear recurrences

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joint work with Xi Chen, Bishal Deb, Alex Dyachenko, and Alan D. Sokal

The Eulerian triangle

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$$H_\infty(\mathbf{a}) = \begin{pmatrix} 1 & 1 & 2 & 6 & \cdots \\ 1 & 2 & 6 & 24 & \cdots \\ 2 & 6 & 24 & 120 & \cdots \\ 6 & 24 & 120 & 720 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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The sequence $\mathbf{a}$ is the **moment sequence** of the **standard exponential distribution**

$$\int_0^\infty x^n e^{-x} dx.$$

Real total positivity

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Coefficientwise total positivity

A matrix with entries belonging to the **polynomial ring** $\mathbb{R}[\mathbf{x}]$ is **coefficientwise totally positive** if all of its minors are polynomials with **nonnegative coefficients** (here $\mathbf{x} = \{x_i\}_{i \geq 0}$ is a possibly infinite set of indeterminates).

Eulerian polynomials

Consider the polynomials

$$A_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$$

where $\left\langle {n \atop k} \right\rangle$ counts **permutations** of $[n]$ with k **descents** (or **ascents**).

$$A = (A_n(x))_{n\geq 0} = (1, 1, x+1, x^2+4x+1, x^3+11x^2+11x+1, \ldots)$$

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$$H_\infty(A) = \begin{pmatrix} 1 & 1 & x+1 & \cdots \\ 1 & x+1 & x^2+4x+1 & \cdots \\ x+1 & x^2+4x+11 & x^3+11x^2+11x+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Theorem

The matrix $H_\infty(A)$ is coefficientwise totally positive.

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$$A = (A_n(x))_{n \geq 0} = (1, 1, x+1, x^2+4x+1, x^3+11x^2+11x+1, \ldots)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Conjecture (Brenti '96)

The matrix

$$A = \left(\left\langle {n+1 \atop k} \right\rangle \right)_{n,k\geq 0}$$

is ***totally positive***.

$$A = \left(\left\langle {n+1 \atop k} \right\rangle \right)_{n,k \geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 11 & 11 & 1 & 0 & \cdots \\ 1 & 26 & 66 & 26 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The “clean” Eulerian triangle (OEIS A008292)

Entries of A satisfy

$$A(n,k) = (n-k+1)A(n-1,k-1) + (k+1)A(n-1,k)$$

for $n \geq 1$ with initial condition $A(0,k) = \delta_{0,k}$.

$$A^{-1}\Delta A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -2 & 7 & 1 & 0 & 0 & \cdots \\ 0 & 14 & -22 & 15 & 1 & 0 & \cdots \\ 0 & -254 & 386 & -154 & 31 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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The **Stirling subset triangle** $S = (S(n,k))_{n,k\geq 0}$ with entries satisfying

$$S(n,k) = S(n-1,k-1) + (k+1)S(n-1,k)$$

for $n \geq 1$ with initial condition $S(0,k) = \delta_{0,k}$.

The **reversed Stirling subset triangle** $S^{rev} = (S^{rev}(n,k))_{n,k\geq 0}$ with entries satisfying

$$S^{rev}(n,k) = (n-k+1)S^{rev}(n-1,k-1) + S^{rev}(n-1,k)$$

for $n \geq 1$ with initial condition $S^{rev}(0,k) = \delta_{0,k}$.

Definition

Let $\boldsymbol{T}(a, c, d, e) = (T(n,k))_{n,k \geq 0}$ be the matrix with entries that satisfy the linear recurrence

$$T(n,k) = [\mathbf{a}(n-k) + \mathbf{c}]\,T(n-1,k-1) + (\mathbf{d}k + \mathbf{e})\,T(n-1,k)$$

for $n \geq 1$ with initial conditions $T(0,k) = \delta_{0k}$.

- Here a, c, d, e are treated as **algebraic indeterminants**.
- The entries of $\boldsymbol{T}(a, c, d, e)$ belong to the **polynomial ring** $\mathbb{Z}[a, c, d, e]$.
- $\boldsymbol{T}(a, c, d, e)$ defines a **family of real lower-triangular matrices**.
- $\boldsymbol{T}(1, 1, 1, 1)$ is the **"clean" Eulerian triangle**.
- $\boldsymbol{T}(0, 1, 1, 1)$ is the **Stirling subset triangle**.
- $\boldsymbol{T}(1, 1, 0, 1)$ is the **reversed Stirling subset triangle**.

Definition

Let $\boldsymbol{T}(a, c, d, e, f, g) = (T(n,k))_{n,k\geq 0}$ be the matrix with entries that satisfy the linear recurrence

$$T(n,k) = [a(n-k)+c]\,T(n-1,k-1) + (dk+e)\,T(n-1,k) + [f(n-2)+g]\,T(n-2,k-1)$$

for $n \geq 1$ with initial conditions $T(0,k) = \delta_{0k}$ and $T(-1,k) = 0$.

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- $\boldsymbol{T}(0, 1, 0, 1, 0, 1)$ is the **Delannoy triangle**.

$$\begin{pmatrix} 1 & 0 & \cdots \\ e & c & \cdots \\ e^2 & ae + c(d+2e) + g & \cdots \\ e^3 & ae(d+3e) + c\left(d^2 + 3de + 3e^2\right) + dg + ef + 2eg & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

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Conjecture (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\boldsymbol{T}(a,c,d,e,f,g)$ *is* **coefficientwise totally positive**.

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Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\boldsymbol{T}(a, c, e) = (T(n,k))_{n,k \geq 0}$ *with entries satisfying*

$$T(n,k) = (\mathbf{a}(n-k) + \mathbf{c})T(n-1, k-1) + \mathbf{e}\,T(n-1, k)$$

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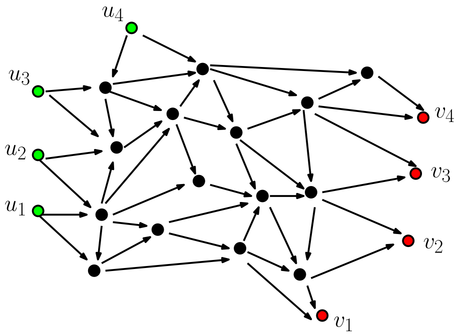
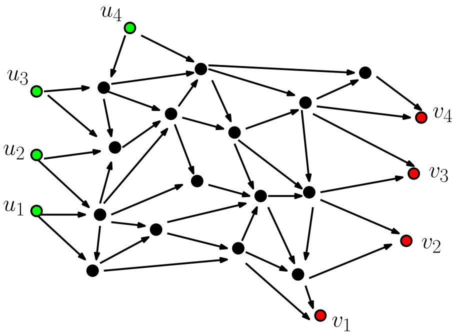
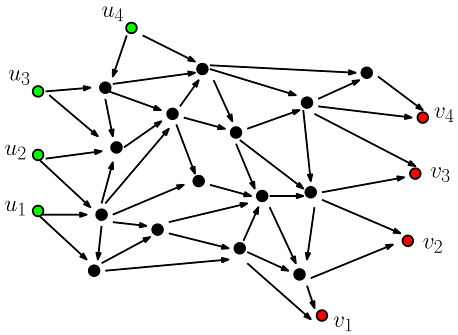
Planar networks



A **locally finite acyclic digraph** D with sources $U = \{u_1, u_2, \ldots, u_n\}$ and sinks $V = \{v_1, v_2, \ldots, v_n\}$.



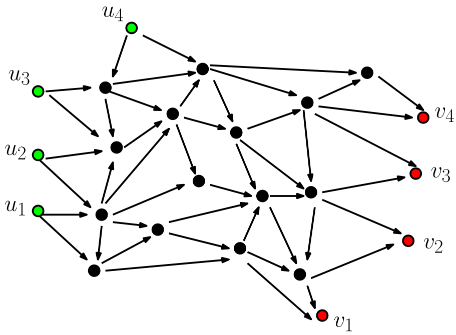
- Each edge $e \in D$ is assigned a weight w_e belonging to some **commutative ring**.
- **Weight of a path** from u_n to v_k: $w(\mathcal{P}) = \prod_{\mathcal{P}:u_n \to v_k} w_e$;
- **Path matrix**: $\boldsymbol{P}_D = (P(u_n \to v_k))_{n,k \geq 1}$ where $P(u_n \to v_k) = \sum_{\mathcal{P}:u_n \to v_k} w(\mathcal{P})$.



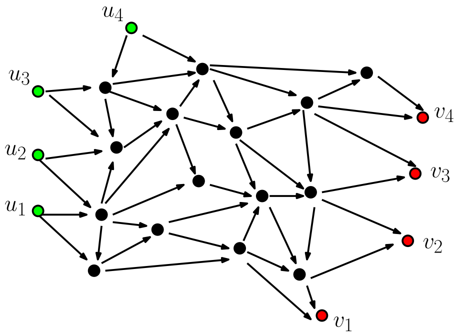
Lemma (KMLGV Lemma)

$$\det(\boldsymbol{P}_D) = \sum_{(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n):U\to V} \operatorname{sgn}(\sigma(\mathcal{P})) \prod_{i=1}^{n} w(\mathcal{P}_i)$$

$\det(\boldsymbol{P}_D)$ gives the **signed sum** over **weighted families of nonintersecting paths**.



U and V are **fully compatible** if for **any subset** of **sources** $u_{n_1}, u_{n_2}, \ldots, u_{n_r}$ (with $n_1 < n_2 < \cdots < n_r$) and **sinks** $v_{k_1}, v_{k_2}, \ldots, v_{k_r}$ (with $k_1 < k_2 < \cdots < v_{k_r}$), the **only** permutation $\sigma \in \mathfrak{S}_r$ mapping u_{n_i} to $v_{k_{\sigma(i)}}$ giving rise to a **nonempty family of nonintersecting paths** is $\sigma = 1$.



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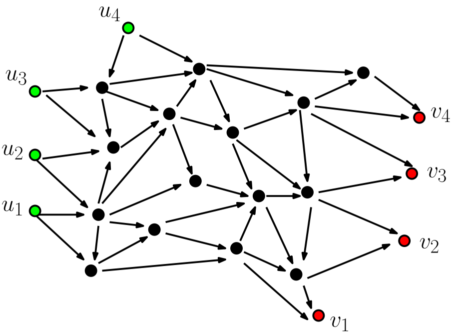
A **planar network** is a **locally finite acyclic digraph** with **fully compatible sources and sinks.**



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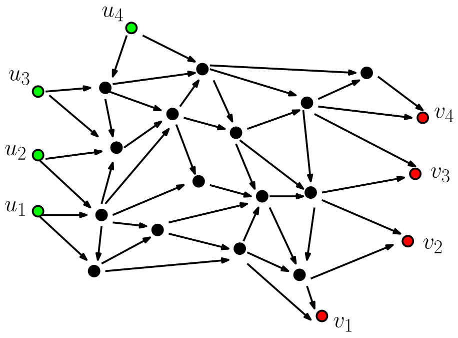
$\det(\boldsymbol{P}_D)$ gives the **signed sum** over **weighted families of nonintersecting paths**.



Lemma (KMLGV Lemma for planar networks)

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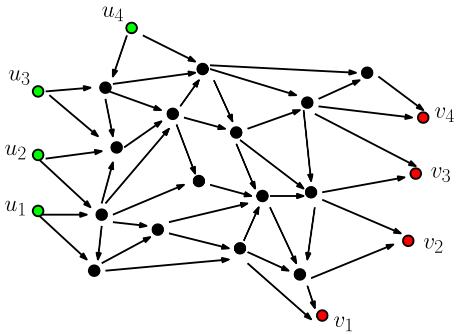
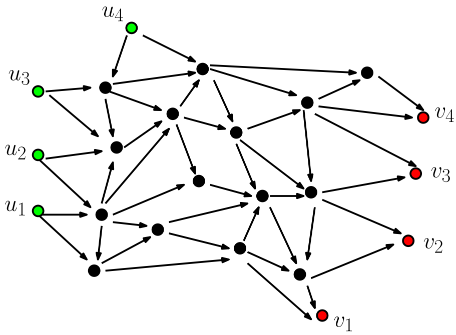


Total positivity and the LGV lemma

- U and V **fully compatible** $\rightarrow$ **every** minor of $\boldsymbol{P}_D$ is a sum over weighted **families of nonintersecting paths**.



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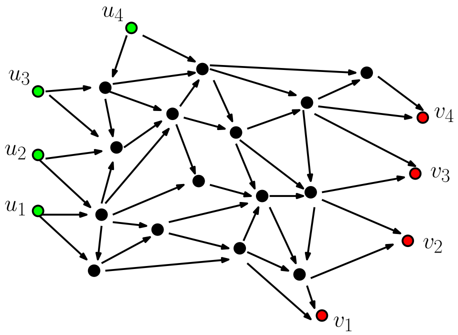
- If $w_e \in \mathbb{R}, w_e \geq 0 \rightarrow \boldsymbol{P}_D$ is **totally positive**;



Total positivity and the LGV lemma

- U and V **fully compatible** $\to$ **every** minor of $\boldsymbol{P}_D$ is a sum over weighted **families of nonintersecting paths**.
- If $w_e \in \mathbb{R}$, $w_e \geq 0 \to \boldsymbol{P}_D$ is **totally positive**;
- If $w_e \in \mathbb{R}[\boldsymbol{x}]$, $w_e \succeq 0 \to \boldsymbol{P}_D$ is **coefficientwise totally positive** in $\mathbb{R}[\boldsymbol{x}]$.

The triangle $\boldsymbol{T}(a, c, e)$

Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\boldsymbol{T}(a,c,e) = (T(n,k))_{n,k\geq 0}$ *with entries satisfying*

$$T(n,k) = (\mathbf{a}(n-k)+\mathbf{c})T(n-1,k-1) + \mathbf{e}\,T(n-1,k)$$

for $n \geq 1$ *with* $T(0,k) = \delta_{0k}$ *is* **coefficientwise totally positive**.

$$\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
\mathbf{e} & \mathbf{c} & 0 & 0 & 0 \\
\mathbf{e}^2 & \mathbf{e}(\mathbf{a}+2\mathbf{c}) & \mathbf{c}^2 & 0 & 0 \\
\mathbf{e}^3 & 3\mathbf{e}^2(\mathbf{a}+\mathbf{c}) & \mathbf{e}\left(\mathbf{a}^2+3\mathbf{ac}+3\mathbf{c}^2\right) & \mathbf{c}^3 & 0 \\
\mathbf{e}^4 & 2\mathbf{e}^3(3\mathbf{a}+2\mathbf{c}) & \mathbf{e}^2\left(7\mathbf{a}^2+12\mathbf{ac}+6\mathbf{c}^2\right) & \mathbf{e}(\mathbf{a}+2\mathbf{c})\left(\mathbf{a}^2+2\mathbf{ac}+2\mathbf{c}^2\right) & \mathbf{c}^4 \\
\vdots & \vdots & \vdots & \vdots & \vdots
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Combinatorial interpretation

The entries of $\boldsymbol{T}(\mathbf{a},\mathbf{c},\mathbf{e})$ satisfy

$$T(n,k) = \sum_{\pi \in \Pi_{n+1,n+1-k}} \prod_{i=2}^{n+1} w_\pi(i)$$

where $\Pi_{n+1,n+1-k}$ is the number of **partitions of** $[n+1]$ into $n+1-k$ **nonempty blocks** and

$$w_\pi(i) = \begin{cases} \mathbf{c} & \text{if smallest}(\pi,i) = 1, \\ \mathbf{e} & \text{if smallest}(\pi,i) = i, \\ \mathbf{a} & \text{if smallest}(\pi,i) \neq i, 1. \end{cases}$$

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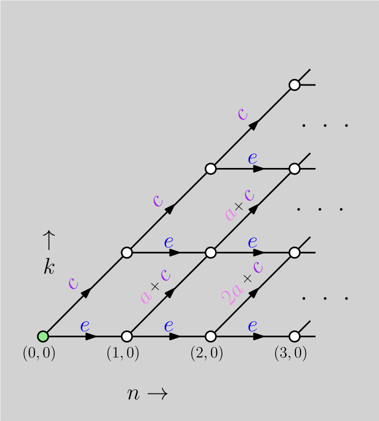

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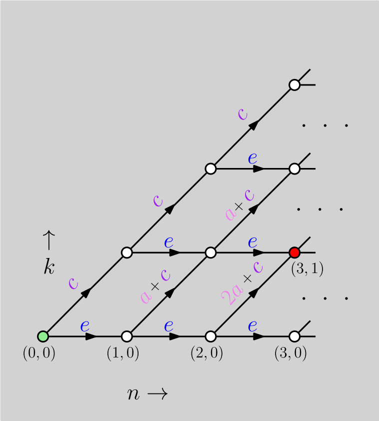

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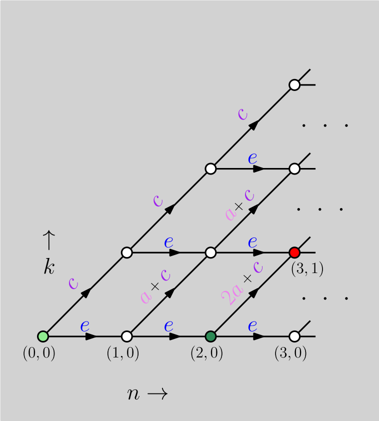

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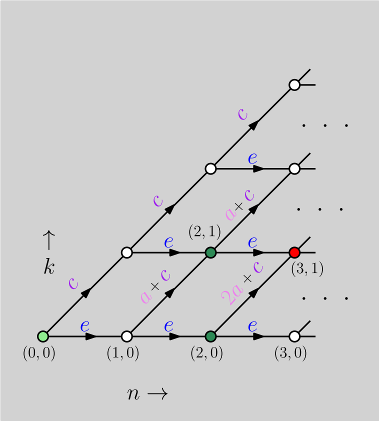

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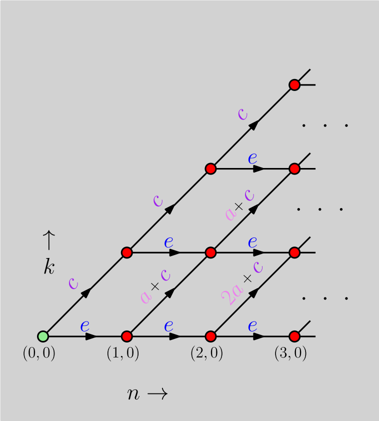

Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\boldsymbol{T}(a, c, e) = (T(n, k))_{n,k \geq 0}$ *with entries satisfying*

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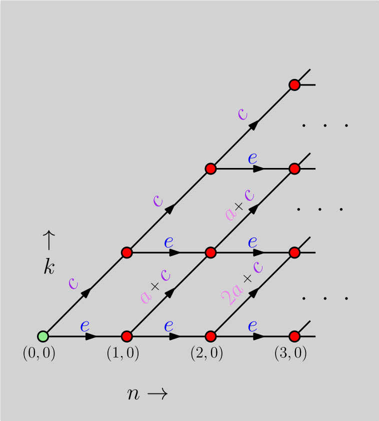

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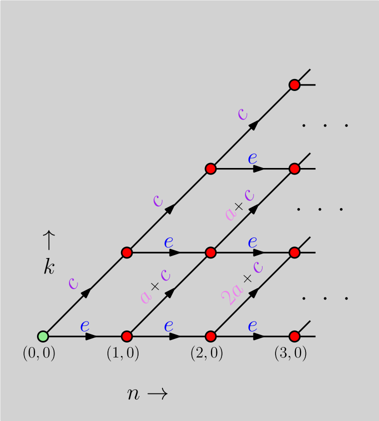

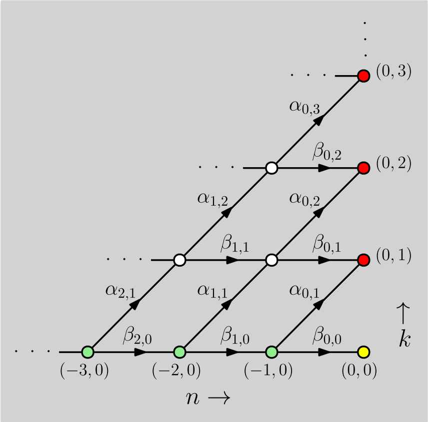

Purely k-dependent recurrence

Consider $T = (T(n,k))_{n,k\geq 0}$ with entries that satisfy

$$T(n,k) = \mathbf{c}\,T(n-1,k-1) + (\mathbf{d}k + \mathbf{e})\,T(n-1,k)$$

for $n \geq 1$ with initial condition $T(0,k) = \delta_{0k}$.

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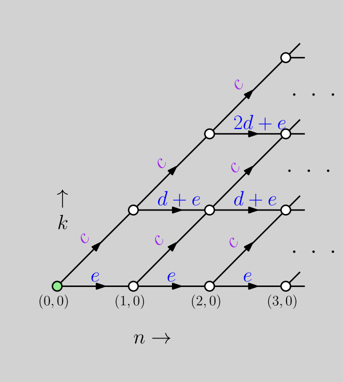
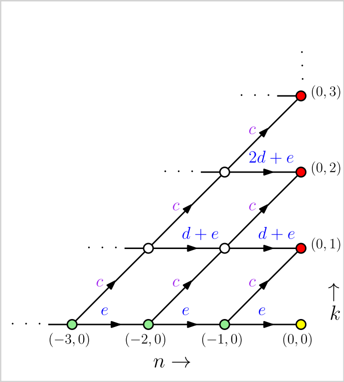


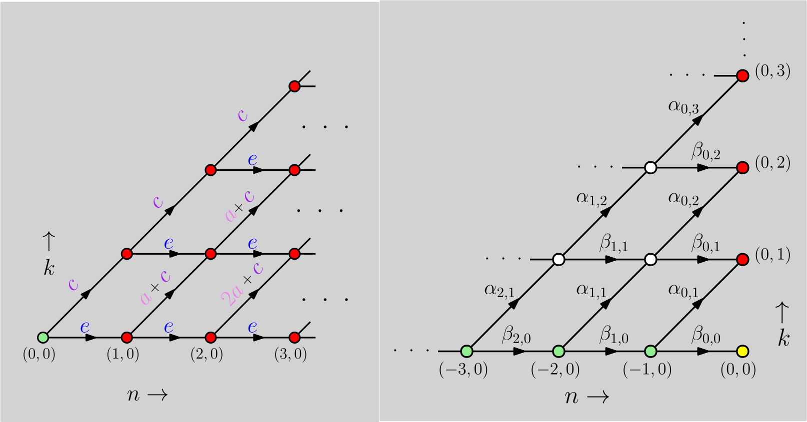


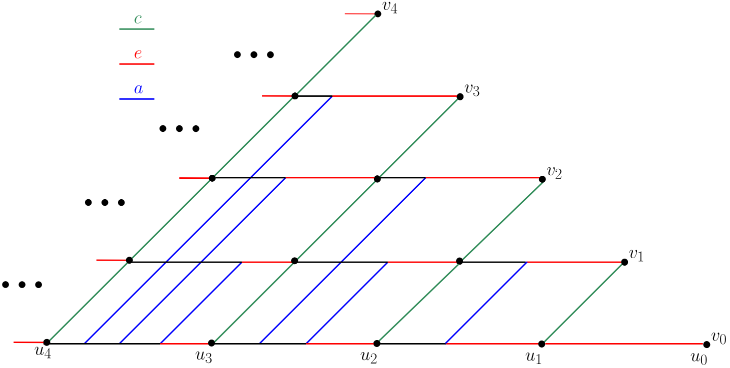
Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

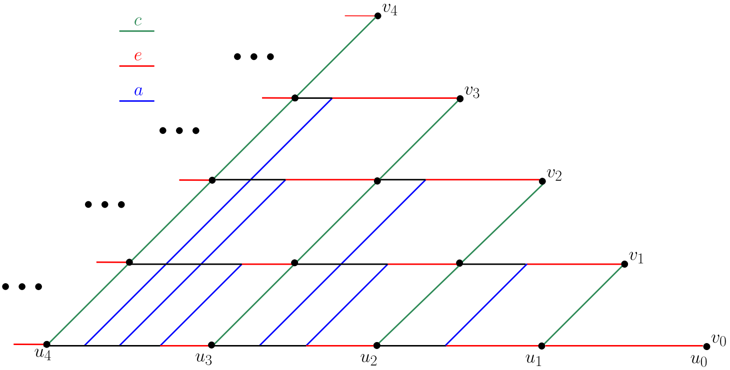
The matrix $\boldsymbol{T}(a, c, e) = (T(n,k))_{n,k\geq 0}$ *with entries satisfying*

$$T(n,k) = (\mathbf{a}(n-k) + \mathbf{c})\,T(n-1,k-1) + \mathbf{e}\,T(n-1,k)$$

for $n \geq 1$ *with* $T(0,k) = \delta_{0k}$ *is* **coefficientwise totally positive**.



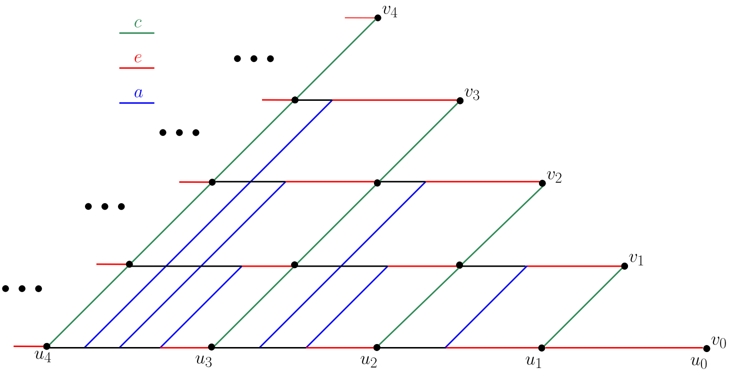

c
e
a
v_4
v_3
v_2
v_1
v_0
u_4
u_3
u_2
u_1
u_0



Entries of $\mathbf{T}(a, c, e) = (T(n, k))_{n,k \geq 0}$ satisfy

$$T(n, k) = (a(n - k) + c)T(n - 1, k - 1) + eT(n - 1, k)$$

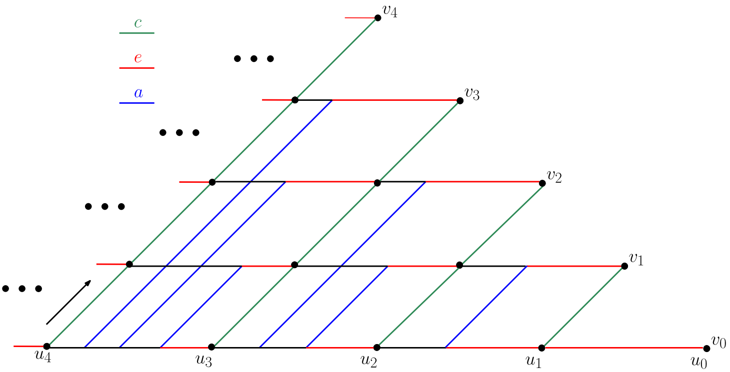
for $n \geq 1$ with initial condition $T(0, k) = \delta_{0,k}$.



The entries of $\mathbf{T}(a, c, e)$ satisfy the **alternative recurrence**:

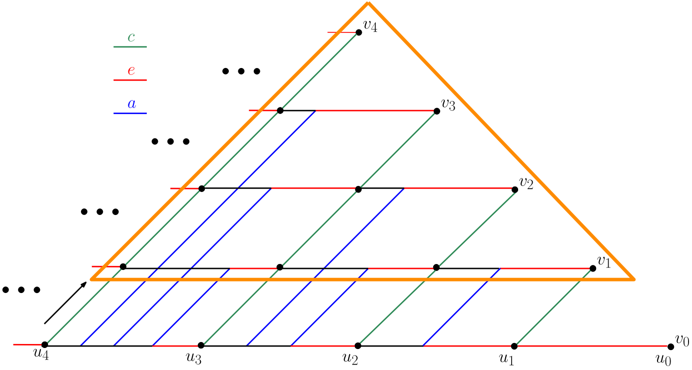
$$T(n, k) = c\,T(n-1, k-1) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e\,T(n-1-m, k-m)$$

for $n \geq 1$, where $T(n, k) = 0$ if $n < 0$ or $k < 0$.



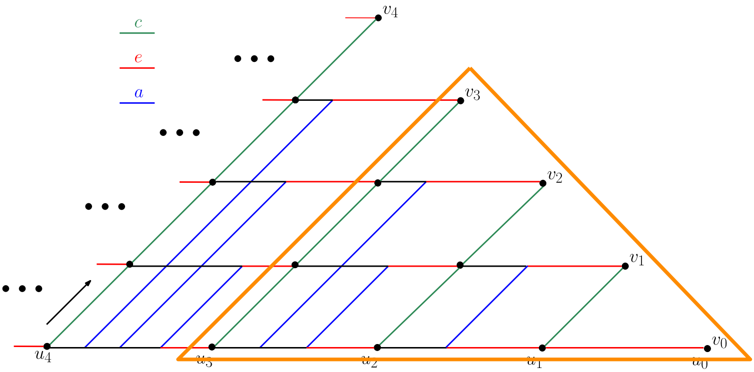
Paths from u_4 to v_k

$$P(u_4 \to v_k) = c$$



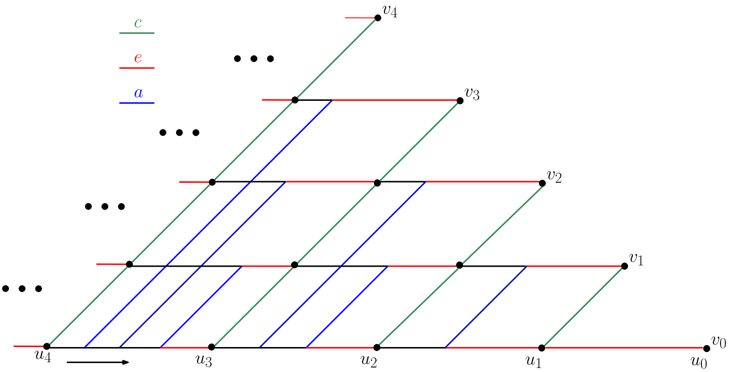
Paths from u_4 to v_k

$$P(u_4 \to v_k) = c$$



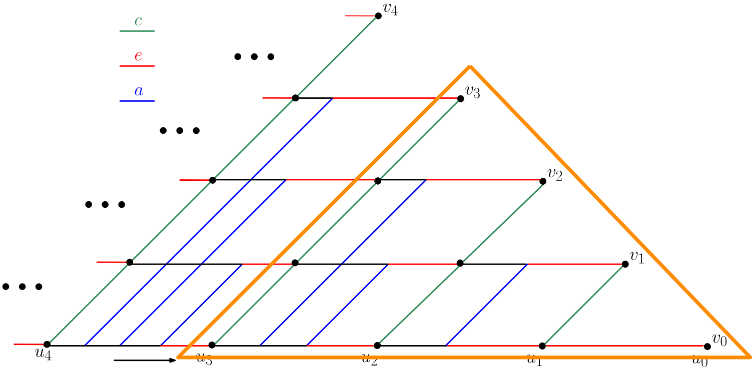
Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) +$$



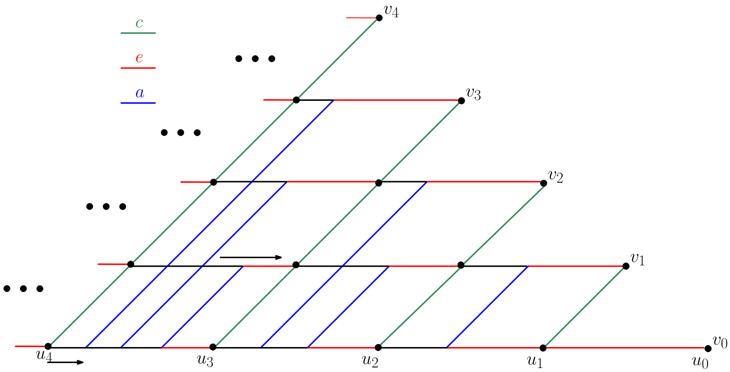
Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} e$$



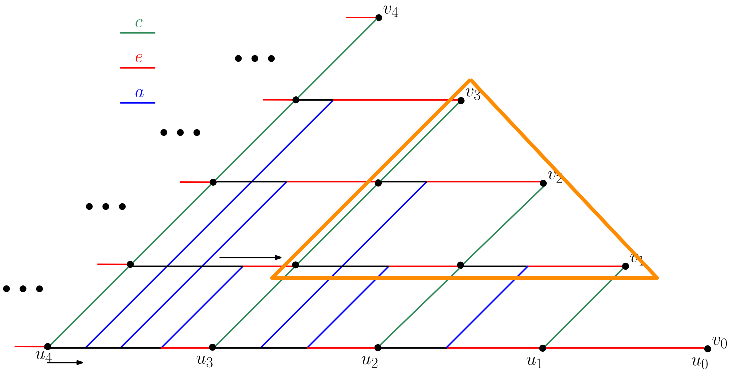
Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$



Paths from u_4 to v_k

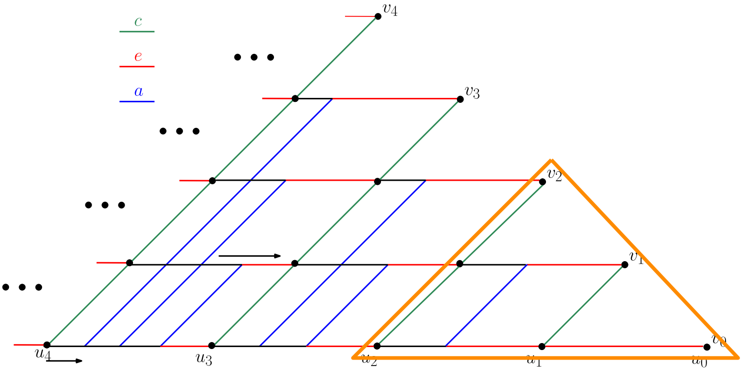
$$P(u_4 \to v_k) = c P(u_3 \to v_{k-1}) + \binom{3}{0} e P(u_3 \to v_k) + \binom{3}{1} ae$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

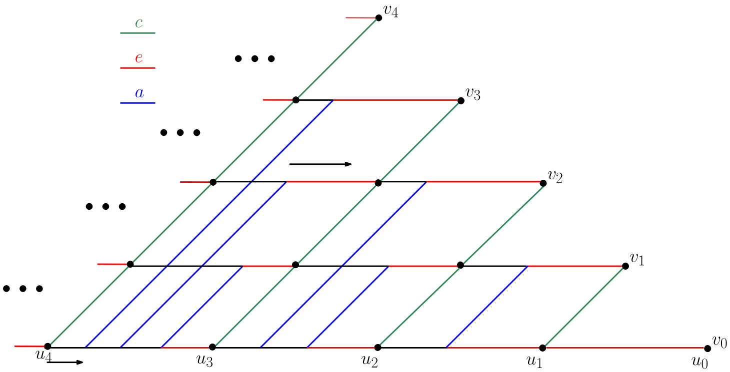
$$\binom{3}{1} ae$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

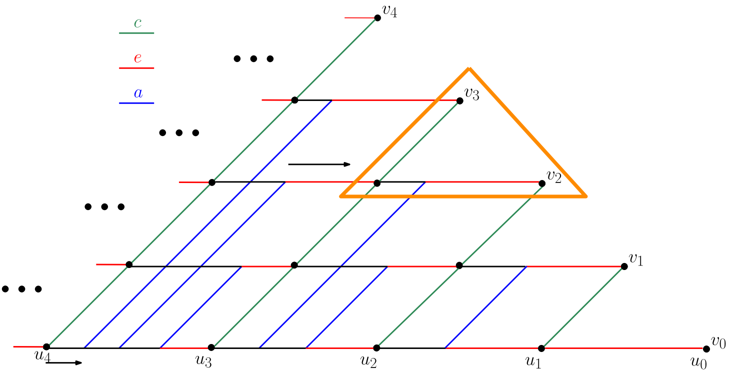
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) +$$



Paths from u_4 to v_k

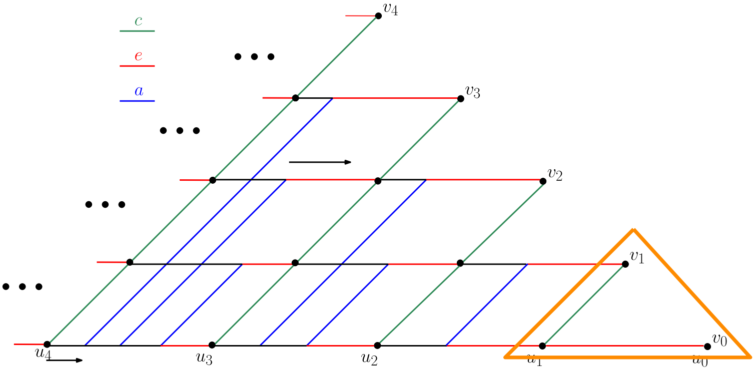
$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

$$\binom{3}{1} aeP(u_2 \to v_{k-1}) +$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) + \binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 e$$



Paths from u_4 to v_k

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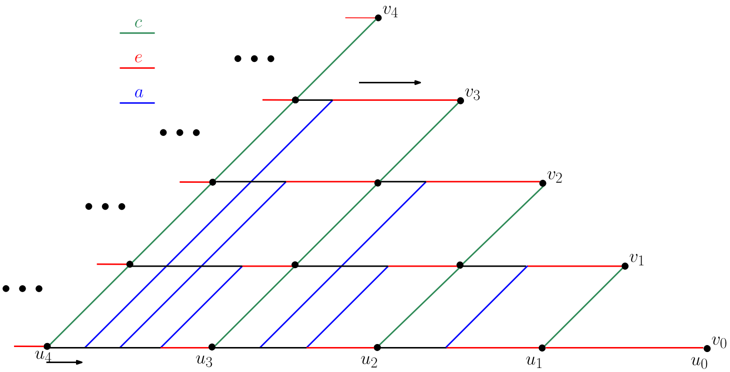
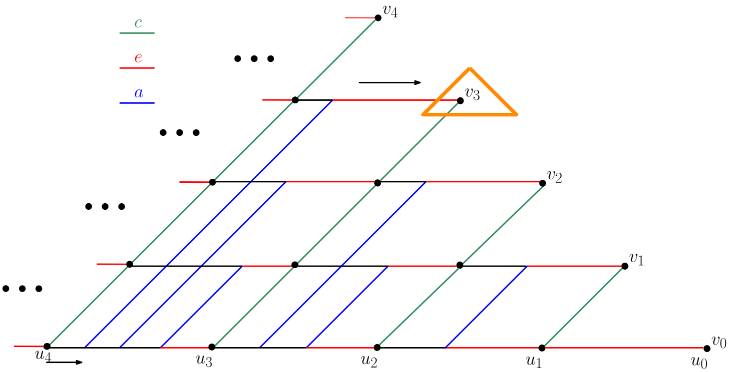


Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

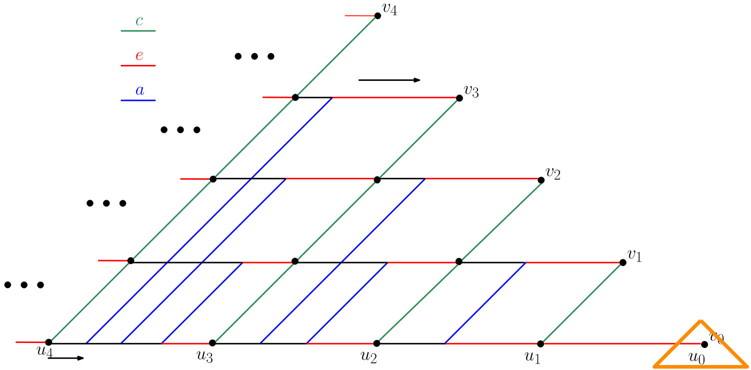
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) +$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

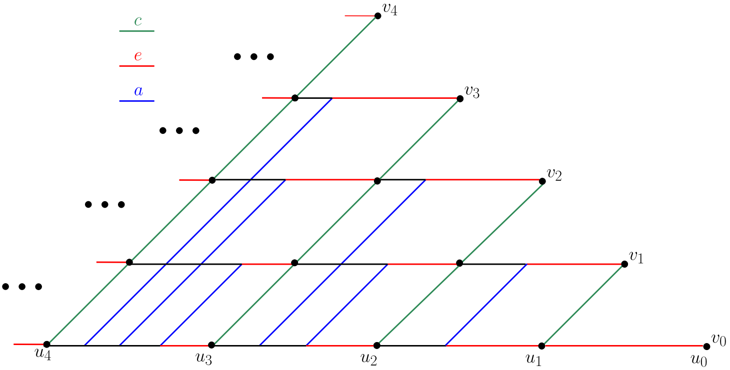
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) + \binom{3}{3} a^3 e$$



Paths from u_4 to v_k

$$P(u_4 \to v_k) = cP(u_3 \to v_{k-1}) + \binom{3}{0} eP(u_3 \to v_k) +$$

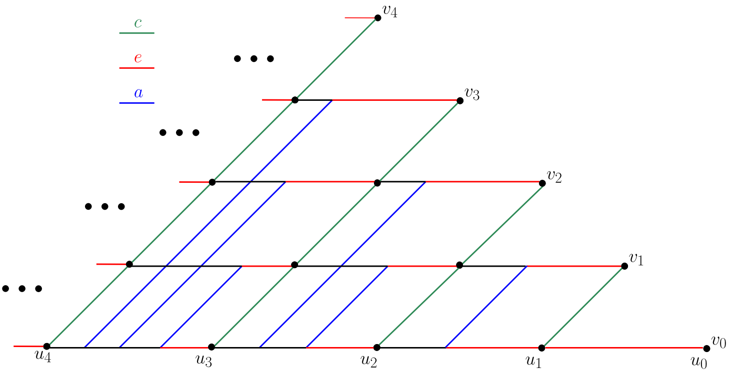
$$\binom{3}{1} aeP(u_2 \to v_{k-1}) + \binom{3}{2} a^2 eP(u_1 \to v_{k-2}) + \binom{3}{3} a^3 eP(u_0 \to v_{k-3})$$



Paths in network satisfy

$$P(u_n \to v_k) = cP(u_{n-1} \to v_{k-1}) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e P(u_{n-1-m}, v_{k-m})$$

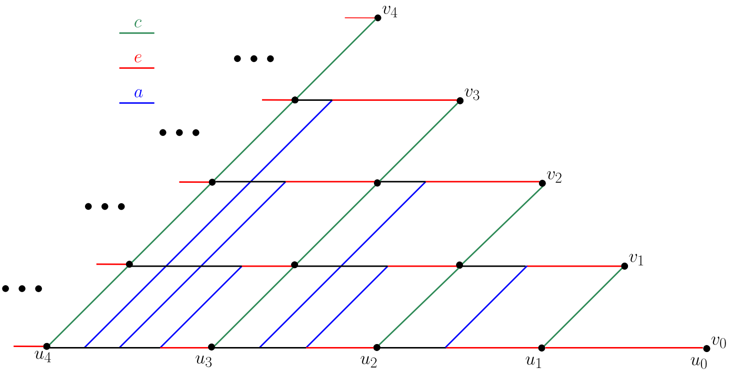
for $n \geq 1$, where $P(u_n \to v_k) = 0$ if $n < 0$ or $k < 0$.



The entries of $\mathbf{T}(a, c, e)$ satisfy the recurrence:

$$T(n,k) = c\,T(n-1,k-1) + \sum_{m=0}^{n-1} \binom{n-1}{m} a^m e\, T(n-1-m, k-m)$$

for $n \geq 1$, where $T(n,k) = 0$ if $n < 0$ or $k < 0$.



Theorem (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\mathbf{T}(a, c, e)$ *is coefficientwise totally positive in the indeterminates* a, c, e.

$$\begin{pmatrix} 1 & 0 & \cdots \\ e & c & \cdots \\ e^2 & ae + c(d+2e) + g & \cdots \\ e^3 & ae(d+3e) + c\left(d^2 + 3de + 3e^2\right) + dg + ef + 2eg & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Conjecture (X. Chen, B. Deb, A. Dyachenko, TG, A. D. Sokal '21)

The matrix $\boldsymbol{T}(a, c, d, e, f, g)$ *is* **coefficientwise totally positive**.

Thank you.