Note: This is a joint work with Volkmar Welker.

ON THE HOMEOMORPHISM AND HOMOTOPY TYPE OF COMPLEXES OF MULTICHAINS

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Let P be a poset with order relation $\leq$. In this paper we study the homeomorphism and homotopy type of simplicial complexes associated to multichains in P. For a number $r \geq 1$ we consider the set P_r of all r-multichains $\mathfrak{p} : p_1 \leq \cdots \leq p_r$ in P.

If $r = 1$ then $P_r = P$ and the order complex $\Delta(P)$ of all linearly ordered subsets of P together with its geometric realization are well studied geometric and topological objects. They have been shown to encode crucial information about P and have important applications in combinatorics and many other fields in mathematics.

In this paper we define for $r \geq 2$ classes of simplicial complexes associated to P_r. In contrast to $r = 1$ there does not seem to be a canonical choice. In [Müh15] a poset structure on P_r is defined whose order complex can be shown to be homotopy equivalent to P. The main focus of our paper is a different construction which leads to a wide class of simplicial complexes associated to P_r. As special cases the construction yields several well studied and important poset and subdivision operations such as interval posets (see [Wal88]), the zig-zag poset (see [PRS98]), the r^{th}-edgewise subdivision (see [EG00]) and a subdivision operation which arose in the work of Cheeger, Müller and Schrader (see [CMS84]), see [Naz21].

For every strictly monotone map $\imath : [r] \to [2r]$ we define a binary relation $\preceq_i$ on P_r. Here for a natural number n we write $[n]$ for $\{1, \ldots, n\}$. Through the undirected graph $G_\imath(P_r) = (P_r, E)$ with edge set

$$E = \big\{\{\mathfrak{p}, \mathfrak{q}\} \subseteq P_r \ : \ \mathfrak{p} \preceq_\imath \mathfrak{q} \text{ and } \mathfrak{p} \neq \mathfrak{q}\big\}$$

we associate to P_r and $\imath$ the clique complex $\Delta(G_\imath(P_r))$ of $G_\imath(P_r)$; that is the simplicial complex of all subsets $A \subseteq P_r$ which form a clique in $G_\imath(P_r)$.

Theorem 0.1. *For $r \geq 2$ the following are equivalent*

(1) *The relation $\preceq_\imath$ is reflexive,*
(2) *The clique complex $\Delta(G_\imath(P_r))$ is a subdivision of $\Delta(P)$.*
(3) *The clique complex $\Delta(G_\imath(P_r))$ is homeomorphic to $\Delta(P)$.*

In the formulation of the theorem and the rest of the paper, we use the term subdivision in the sense of geometric subdivision (see e.g. [Sta92]). Also when we speak of homotopy equivalent or homeomorphic simplicial complexes we mean that their geometric realizations are homotopy equivalent or homeomorphic.

The cases treated in the theorem include the above mentioned subdivision operations as special cases. Indeed, we show that for fixed r there are exactly 2^{r-1} different $\imath$ for which $\preceq_\imath$ is reflexive and $\imath(1) = 1$. We will see that the latter condition eliminates an obvious symmetry.

If $\preceq_\imath$ is even a partial order then $\Delta(G_\imath(P_r))$ coincides with the order complex of P_r with respect to $\preceq_\imath$. The following proposition shows that for each r there is exactly one $\imath$ for which $\preceq_\imath$ is a partial order.

Proposition 0.2. *The relation $\preceq_\imath$ is a partial order on P_r if and only if for $2 \leq t \leq r$ we have* $\imath(t) = \begin{cases} 2t, & t \text{ is even;} \\ 2t-1, & t \text{ is odd.} \end{cases}$

For the $\imath$ from the above proposition and $r = 2$ we have that $\preceq_\imath$ coincides with the interval order on P and for arbitrary $r \geq 2$ we have that $\preceq_\imath$ defines the zig-zag order on P_r (see [PRS98]). It can be seen (see [Naz21]) that the order complex of the zig-zag order on P_r coincides for even r with a subdivision operation studied in [CMS84].

The condition that $\preceq_\imath$ is reflexive provides the most restrictions on $\imath$ when classifying the $\preceq_\imath$ which are partial orders. Thus instead of the relation $\preceq_\imath$, we may consider the relation $\preceq'_\imath$ which is defined as follows:

$$\mathfrak{p} \preceq'_\imath \mathfrak{q} :\Leftrightarrow \begin{cases} \mathfrak{p} = \mathfrak{q} \text{ or} \\ \mathfrak{p} \neq \mathfrak{q} \text{ and } \mathfrak{p} \preceq_\imath \mathfrak{q} \end{cases}$$

We classify when $\preceq'_\imath$ is partial order and show the following theorem.

Theorem 0.3. *If $\preceq'_\imath$ is a partial order then the order complex of P_r with respect to $\preceq'_\imath$ is homotopy equivalent to $\Delta(P)$.*

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